Sparse Polynomials in Polynomial Ideals

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Introduction

Let K be a field, $K[x_0, \dots, x_n]$ a polynomial ring and $I \subseteq K[x_0, \dots, x_n]$ an ideal. The *support* of a polynomial $f = \sum_\alpha f_\alpha x^\alpha$ is

$$\operatorname{supp}(f) := \{\alpha : f_\alpha \neq 0\}.$$

It is a natural question to ask what polynomials in an ideal have the smallest supports, i.e. the smallest numbers of summands. More generally, we might be interested in the *circuits* of an ideal, which are the polynomials with inclusion-minimal support.

Syzygies, i.e. relations between the generators of an ideal, encode lots of information but not the supports. As an example, a principal ideal does not have any syzygies, although the behaviour of the sparsest polynomials can vary. Consider the ideals $I := \langle x^2 + xy + y^2 \rangle$ and $J := \langle x^2 + y^2 \rangle$ in $K[x, y]$. Then I contains a polynomial with fewer sumands than its generator, namely $(x-y)(x^2+xy+y^2) = x^3 - y^3$. But J does not, which can be seen as follows. The *Newton polytope* of a polynomial is the convex hull of its support. Therefore, the Newton polytope of $x^2 + y^2$ is a line segment. The Newton polytope of every other polynomial in J is the Minkowski sum of the Newton polytope of $x^2 + y^2$ with another polytope. Since the number of vertices cannot decrease under that operation, every Newton polytope of an element in J has to have at least two vertices. Thus every polynomial in J has to have at least two summands.

Apart from syzygies, other classical tools, like primary decomposition or the Castelnouvo-Mumford regularity, seem not suitable for this problem, either. Consider the following example given by Jensen, Kahle, Katthän [**1**]. For any $n \in \mathbb{N}$, let $I_n = \langle (x-z)^2, nx - y - (n-1)z \rangle \subseteq \mathbb{Q}[x, y, z]$. The ideals I_n all have Castelnouvo-Mumford regularity 2 and are primary over $\langle x - z, y - z \rangle$. Furthermore, the binomial $x^n - yz^{n-1}$ is containend in I_n. It can be shown that there is no binomial with degree less than n in that ideal.

This bachelor thesis develops two approaches to study short polynomials in a polynomial ideal. In Chapter 1, we introduce necessary notions from Commutative Algebra and Algebraic Geometry. Most notably, this includes homogeneous ideals and their projective geometry.

The first approach is developed in Chapter 2. It is motivated by Lemma 27, which is a special case of matroid duality. We can derive information on supports of vectors in a K-vector space $V \subseteq K^n$, which orthogonal complement is given as the rowspace of a matrix M, by looking at the rank of certain submatrices of M. But polynomials can detect the rank of a matrix, so we reformulate this statement in terms of Algebraic Geometry. This leads to Theorem 40, which basically says

that if the Hilbert function of an homogeneous ideal I is known, it is a polynomial condition on the coefficients of generators of I whether a certain homogeneous component contains a nonzero polynomial with a small number of summands.
The second part of this text starts off with Chapter 3, which focuses on determinantal ideals and how they provide a bridge between Commutative Algebra and Combinatorics.
In Chapter 4, we explain our second approach to study short polynomials in polynomial ideals, by tackling the following question. Given generators of an ideal, can we calculate how short the polynomials in that ideal can get? The key observation here is Lemma 69, which links our question of interest to the dimensions of certain vector spaces. We develop our approach in the particular case of classical determinantal ideals, which leads to the following concrete result. If I_t denotes the determinantal ideal generated by t-minors of a generic matrix over a field K, it does not contain a nonzero polynomial with at most $\frac{t!}{2}$ summands. However, I think that this bound can be improved. The reason for my hope relies on the proof and is explained at the end of this bachelor thesis.

Notation and conventions

The symbol $\mathbb{N}$ denotes the natural numbers without zero and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We define $|a| := \sum_{i=1}^{n} a_i$ for any tuple $a = (a_1, \dots, a_n) \in \mathbb{N}_0^n$ and the set $\mathbb{N}_t^n := \{\alpha \in \mathbb{N}_0^n | |\alpha| = t\}$ for every $t \in \mathbb{N}$.
For any $n \in \mathbb{N}$, we denote the set $\{1, \dots, n\}$ by $[n]$.
Let X be a set and consider a matrix $M \in K^{m \times n}$ with entries in X, as well as row and column labels I, J. Then $M_{*,J}$ denotes the submatrix of M with all rows and columns indexed by J, $M_{I,*}$ denotes the submatrix of M with rows indexed by I and all columns, and $M_{I,J}$ denotes the submatrix of M with rows indexed by I and columns indexed by J.
For any two sets X, Y, we define X^Y to be the set of families $(x_y)_{y \in Y} \subseteq X$.

CHAPTER 1

Commutative Algebra and Algebraic Geometry

In this chapter, we collect basic algebraic and geometric notions for this text. In particular, this includes homogeneous polynomial ideals and their projective geometry. Most definitions in this chapter are special cases of more general notions. For instance, we only define ideals for polynomial rings with a finite number of variables over a field. This and similar restrictions were made, because they are sufficient for later considerations. Sometimes, we follow the reference [**2**]. Since this book is not officially published yet, we will always substantiate its content using other standard sources. These are given in the corresponding sections.

1. Ideals

Let us fix, once and for all, a field K. We denote its polynomial ring in $n+1$ variables by $K[x_0, \ldots, x_n]$, although we might use different letters when n is small. For example, we may write $K[x, y, z]$ when $n = 2$.
The following definition is fundamental.

DEFINITION 1. *A subset I of $K[x_0, \ldots, x_n]$ is called* (polynomial) ideal*, if the following conditions hold.*

- $f, g \in I$ *implies* $f + g \in I$
- $h \in K[x_0, \ldots, x_n]$ *and* $f \in I$ *implies* $h \cdot f \in I$

Ideals are the ring-theoretic analogues of normal subgroups. For instance, one can show that ideals are exactly the kernels of ring homomorphisms. The proof of this characterisation uses the construction of *quotient rings* $K[x_0, \ldots, x_n]/I$, which are the ring-theoretic analogues of quotient groups.
Another motivation for the definition of ideals comes from Algebraic Geometry. For every subset of the *affine space* $\mathbb{A}^{n+1} := K^{n+1}$, the set of polynomials in $K[x_0, \ldots, x_n]$ vanishing on this set is an ideal. This connection is explained more deeply in the case of projective space $\mathbb{P}^n$ in Section 2.

We are already able to give first examples of ideals. The *zero ideal* $\{0\}$ and the whole ring $K[x_0, \ldots, x_n]$ are called *trivial ideals* of $K[x_0, \ldots, x_n]$. To give a non-trivial example, fix a polynomial $p \in K[x_0, \ldots, x_n]$. Then the set I of polynomials $f \in K[x_0, \ldots, x_n]$ divisible by p is an ideal.

Proof. Let $f, g \in I$ and $h \in K[x_0, \ldots, x_n]$. Then we can write $f = f'p$ and $g = g'p$, where $f', g' \in K[x_0, \ldots, x_n]$. The calculations

$$f + g = f'p + g'p = (f' + g')p \in I$$

and

$$hf = h(f'p) = (hf')p \in I$$

show that I is indeed an ideal. □

Definition 2. *An ideal* $I \subseteq K[x_0, \dots, x_n]$ *is* prime *if for every polynomials* $f, g \in K[x_0, \dots, x_n]$ *such that its product* fg *is in* I*, we have either* $f \in I$ *or* $g \in I$*.*

As an example, let $K[x_0, \dots, x_n] = K[x]$ be the polynomial ring in one variable, I_1 the ideal of polynomials divisible by x and I_2 the ideal of polynomials divisible by x^2. Then I_1 is prime, because if $fg \in I_1$, either f or g has to be divisible by x, But I_2 is not, because although $x \cdot x \in I_2$, we have $x \notin I_2$.

1.1. Generating sets. We already saw first examples of ideals. Unfortunately, speaking of "the ideal of polynomials divisible by a fixed polynomial p" is not very efficient. This problem can be solved using generating sets.

Definition 3. *Let* M *be a set of polynomials in* $K[x_0, \dots, x_n]$*. The* ideal generated by M *is*

$$\langle M \rangle := \bigcap_{\substack{I \subseteq K[x_0, \dots, x_n] \text{ ideal} \\ M \subseteq I}} I$$

If $M = \{m_1, \dots, m_s\}$ *is finite, we write*

$$\langle M \rangle = \langle m_1, \dots, m_s \rangle.$$

It is straightforward to check that $\langle M \rangle$ is indeed an ideal.
With this definition, the ideal I of polynomials $f \in K[x_0, \dots, x_n]$ divisible by a fixed polynomial $p \in K[x_0, \dots, x_n]$ is just $I = \langle p \rangle$. This oberservation is generalized by the following lemma.

Lemma 4. *For any* $M \subseteq K[x_0, \dots, x_n]$*, we have*

$$\langle M \rangle = \left\{ \sum_{i=1}^{s} f_i m_i \colon f_1, \dots, f_s \in K[x_0, \dots, x_n], m_1, \dots, m_s \in M \right\}.$$

Proof. Let I denote the set on the left hand side of the equation above. Obviously, I is an ideal which contains M. This proves the inclusion $\langle M \rangle \subseteq I$. To prove the converse, let $J \subseteq K[x_0, \dots, x_n]$ be any ideal such that $M \subseteq J$. We have to show $I \subseteq J$. Let $\sum_{i=1}^{s} f_i m_i \in I$, then we also have $m_1, \dots, m_s \in J$. This implies $f_1 m_1, \dots, f_s m_s \in J$ and thus $\sum_{i=1}^{s} f_i m_i \in J$. □

We'd like to answer two natural questions about generating sets. First, which ideals admit a finite generating set? Second, is the generating set of an ideal unique?

The first question is answered by *Hilbert's Basis Theorem.* The general formulation can be found in [**3**].

Theorem 5 (Hilbert Basis Theorem). *Every ideal in a polynomial ring over a field* K *has a finite generating set.*

The answer to the second question is no. For example, in $K[x]$ we have

$$\langle x \rangle = \langle x^2 + x, x^2 \rangle.$$

However, we could ask whether an ideal admits a preferable generating set. An answer to this question is given by the theory of Gröbner bases to which we briefly introduce. We follow [**4**, Chapter 1] and [**3**, Section 15].

A total order $\prec$ on $\mathbb{N}_0^{n+1}$ is a *term order*, if $0 \in \mathbb{N}_0^{n+1}$ is the unique minimal element and for elements $a, b \in \mathbb{N}_0^{n+1}$ with $a \prec b$ it follows that $a + c \prec b + c$ for all $c \in \mathbb{N}_0^{n+1}$. A standard example is the *lexicographic ordering*: we set $a \prec_{lex} b$ if $a_i < b_i$ for the first index i such that $a_i \neq b_i$.

A *monomial* in the polynomial ring $K[x_0, \dots, x_n]$ is a product of variables $x^a := x_0^{a_0} \cdots x_n^{a_n}$, where $x = (x_0, \dots, x_n)$ and $a = (a_0, \dots, a_n) \in \mathbb{N}_0^{n+1}$. The identification between monomials of $K[x_0, \dots, x_n]$ and elements of $\mathbb{N}_0^n$ enables us to view a term order $\prec$ as an order on the monomials.

For a polynomial $f \in K[x_0, \dots, x_n]$, we define its *initial monomial* $in_\prec(f)$. Let $f = \sum_{\alpha \in \mathbb{N}_0^{n+1}} f_\alpha x^\alpha$, then $in_\prec(f) := f_\beta x^\beta$ if $x^\beta = \max_{\alpha \in \operatorname{supp}(f)} x^\alpha$ under the term order $\prec$. Moreover, every ideal $I \subseteq K[x_0, \dots, x_n]$ has an *initial ideal*

$$in_\prec(I) := \langle in_\prec(f) \colon f \in I \rangle.$$

The monomials not in $in_\prec(I)$ are called *standard monomials* and have the following property, which can be found in [**4**, Proposition 1.1].

Proposition 6. *The images of the standard monomials form a K-vector space basis for $K[x_0, \dots, x_n]/I$.*

We can now give the following definition.

Definition 7. *A finite subset $\mathcal{G} \subseteq I$ of an ideal I is* Gröbner basis *with respect to a term order $\prec$ if the initial ideal $in_\prec(I)$ is generated by $\{in_\prec(g) \colon g \in \mathcal{G}\}$. Moreover, a Gröbner basis $\mathcal{G}$ is* reduced*, if the leading coefficient for every polynomial in $\mathcal{G}$ is 1 and for every two distinct polynomials $g_1, g_2 \in \mathcal{G}$, no term of g_2 is divisible by $in_\prec(g_1)$.*

Although it is not obvious from the definition, a Gröbner basis is a generating set for the corresponding ideal. Moreover, it can be shown that every ideal admits a reduced Gröbner Basis which is unique for a fixed term order.

With regards to our motivation to the theory of Gröbner bases, a reduced Gröbner basis for an ideal is in a sense preferable to other generating sets. For example, starting with any set of generators for an ideal I, one can compute the reduced Gröbner basis $\mathcal{G}$ for a given term order using *Buchberger's algorithm.* Also, it is possible to solve the *ideal membership problem*: Given a polynomial $f \in K[x_0, \dots, x_n]$ and an ideal $I \subseteq K[x_0, \dots, x_n]$, decide whether $f \in I$. This is done using the *division algorithm.*

We close this section with a connection between the Gröbner basis and the circuits of an ideal.

Definition 8. *The* support *of a polynomial $f = \sum_\alpha f_\alpha x^\alpha$ is*

$$\operatorname{supp}(f) := \{\alpha \colon f_\alpha \neq 0\}.$$

A circuit *of an ideal $I \subseteq K[x_0, \dots, x_n]$ is an inclusion-minimal element of the set of supports of polynomials in I.*

The following lemma and its proof are almost copied from [**5**, Lemma 4.1.4], where a similar result is proven in a different setting. Unfortunately, the argument for proving the converse does not work in our case.

Lemma 9. *Let $\mathcal{G}$ be a reduced Gröbner Basis of an ideal $I \subseteq K[x_0, \dots, x_n]$. Then the support of every $g \in \mathcal{G}$ is a circuit of I.*

Proof. Let $g = \sum_{\alpha \in S} g_\alpha x^\alpha \in \mathcal{G}$ such that $S = \operatorname{supp}(g)$ and let $\mathcal{G}$ be a reduced Gröbner basis for the term order $\prec$. Suppose that S is not a circuit of I. This

means that there exists a polynomial $f = \sum_{\alpha \in S} f_\alpha x^\alpha \in I$ with some $f_\beta = 0, \beta \in S$. Let $x^\gamma = in_\prec(g)$. We claim that $x^\alpha \notin in_\prec(I)$ for every $\alpha \in S \setminus \{\gamma\}$. Suppose there exists an element $\alpha \in S \setminus \{\gamma\}$ such that $x^\alpha \in in_\prec(I)$. Since $in_\prec(I)$ is generated by $\{in_\prec(g)\colon g \in \mathcal{G}\}$, x^α is divisible by $in_\prec(\tilde{g})$ for some $\tilde{g} \in \mathcal{G}$. Since $\mathcal{G}$ is reduced it follows that $\tilde{g} = g$ and thus $in_\prec(g) = x^\gamma$ divides x^α. This means that there exists a δ such that $x^\delta x^\gamma = x^\alpha$ which implies $\delta + \gamma = \alpha$. Now we have $\gamma \succ \alpha$, therefore $\delta + \gamma \prec \gamma$ and thus $\delta \prec 0$. Since for a term order, 0 is the unique minimal element, is follows that $\delta = 0$ and thus $\gamma = \alpha$. But this is a contradiction, because $\alpha \in S \setminus \{\gamma\}$. This proves the claim.
Because of Proposition 6 there is no polynomial in I with support in $S \setminus \{\gamma\}$. It follows that $f_\gamma \neq 0$. But then $g - f_\gamma^{-1} \sum_{\alpha \in S} f_\alpha x^\alpha$ lies in I, which is a contradiction. □

1.2. Homogeneous ideals. We give a short introduction to homogeneous ideals, following [**6**, Section I.2].

Definition 10. *A polynomial* $f \in K[x_0, \ldots, x_n]$ *is* homogeneous of degree $d \in \mathbb{N}_0$ *if* $\mathrm{supp}(f) \subseteq \mathbb{N}_d^{n+1}$.

As an example, consider the polynomials $f, g \in K[x, y]$ given by $f = x^2y - y^3$ and $g = x^2y - y^2$. Then f is homogeneous, but g is not.
We can extend the previous definition to ideals.

Definition 11. *An ideal I is* homogeneous *if it is generated by homogeneous polynomials.*

It turns out that there is a second characterisation of homogeneous ideals, which we explain now. First, we define the following vector spaces. For any natural number $d \in \mathbb{N}_0$ the *d-th homogeneous component of* $K[x_1, \ldots, x_n]$ is

$$K[x_0, \ldots, x_n]^{(d)} := \{f \in K[x_0, \ldots, x_n]\colon\ f \text{ homogeneous of degree } d\} \cup \{0\}.$$

This composes the polynomial ring as a vector space

$$K[x_0, \ldots, x_n] = \bigoplus_{d=0}^{\infty} K[x_0, \ldots, x_n]^{(d)}$$

such that

$$K[x_0, \ldots, x_n]^{(d)} \cdot K[x_0, \ldots, x_n]^{(e)} \subseteq K[x_0, \ldots, x_n]^{(d+e)}$$

for any $d, e \in \mathbb{N}_0$. Note that $K[x_0, \ldots, x_n]^{(d)}$ has finite dimension: the monomials of degree d form a K-basis.
This construction makes $K[x_0, \ldots, x_n]$ a *graded ring*. Graded rings provide a general framework for homogeneous ideals. But since we are only concerend with polynomial ideals, we will continue working in this special case.

We extend the notion of homogeneous components to ideals.

Definition 12. *Let* $I \subseteq K[x_0, \ldots, x_n]$ *be an ideal, then its* d-th homogeneous component *is the K-vector space*

$$I^{(d)} := I \cap K[x_0, \ldots, x_n]^{(d)}.$$

We only use the notion of homogeneous components for homogeneous ideals, because it is well-behaved in this case.

LEMMA 13. *An ideal $I \subseteq K[x_0, \dots, x_n]$ is homogeneous if and only*

$$I = \bigoplus_{d=0}^{\infty} I^{(d)}$$

as K-vector spaces.

Note that the previous Lemma provides a second characterisation of homogeneous ideals, as promised above.
We are now able to define an important invariant. Let $I \subseteq K[x_0, \dots, x_n]$ be homogeneous. For any $d \in \mathbb{N}_0$, the dimension of $I^{(d)}$ is finite, since it is a subspace of $K[x_0, \dots, x_n]^{(d)}$. Thus we can define its *Hilbert function*

$$\begin{aligned} H_I \colon \mathbb{N}_0 &\to \mathbb{N}_0 \\ d &\mapsto \dim_K I^{(d)}. \end{aligned}$$

A priori, one might think that the Hilbert function of a polynomial ideal behaves messy, but this is not true. For any ideal I exists a polynomial P_I such that $P_I(d) = H_I(d)$ for d sufficiently large. Further information can be found in [**7**].

2. Projective geometry

We mentionend that ideals appear naturally in Algebraic Geometry. We explain this relationship, following the references [**8**, Section 4] and [**6**, Section I.2].

2.1. Definitions. The ambient space for projective geometry is the projective space.

DEFINITION 14. *We define an equivalence relation on the set $V \setminus \{0\}$ for any finite dimensional K-vector space V as follows. We set $v \sim v'$ if and only if there exists a $\lambda \in K \setminus \{0\}$, such that $\lambda v = v'$. The* projective space $\mathbb{P}(V)$ *is the set of equivalence classes under this equivalence relation. We denote the equivalence class of an element $v \in V \setminus \{0\}$ by $[v]$. If we fix a basis $K^{n+1} \cong V$, we call $\mathbb{P}^n := \mathbb{P}(K^{n+1})$* n-dimensional projective space over *K and denote the equivalence class of a vector $(a_0, \dots, a_n) \in K^{n+1}$ by $[a_0 : \dots : a_n]$.*

The *Nullstellensatz* provides a strong relationship between homogeneous ideals in $K[x_0, \dots, x_n]$ and geometry in $\mathbb{P}^n$. However, we will not give the whole statement, but at least describe some related concepts. A homogeneous polynomial $f \in K[x_0, \dots, x_n]$ does not define a function $\mathbb{P}^n \to K$. This can be seen, for example, by looking at the case $n = 0$ and $f = x_0$. But it is possible to talk about zeros of f in $\mathbb{P}^n$.

LEMMA 15. *Let $(a_0, \dots, a_n) \sim (a_0', \dots, a_n')$ and $f \in K[x_0, \dots, x_n]$ homogeneous. Then $f(a_0, \dots, a_n) = 0$ if and only if $f(a_0', \dots, a_n') = 0$.*

Proof. Let $d = \deg f$, $(a_0, \dots, a_n) = \lambda(a_0', \dots, a_n')$ for a $\lambda \in K \setminus \{0\}$ and suppose that $f(a_0', \dots, a_n') = 0$. Since f is homogeneous, we have

$$\begin{aligned} f(a_0, \dots, a_n) &= f(\lambda a_0', \dots, \lambda a_n') \\ &= \lambda^d f(a_0', \dots, a_n') \\ &= 0. \end{aligned}$$

The converse follows similarly. ∎

Lemma 15 enables us to say that a $p \in \mathbb{P}^n$ is a zero of an homogeneous polynomial f, by choosing an arbitrary element $(a_0, \ldots, a_n) \in p$ and evaluating $f(a_0, \ldots, a_n)$. This leads to the following definition.

Definition 16. *The* zero set *of a set of homogeneous polynomials T in a polynomial ring is*

$$Z(T) := \{p \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in T\}.$$

We are now able to define a topology on a projective space.

Definition 17. *We define the* Zariski topology *on $\mathbb{P}^n$ by taking the closed sets to be the zero sets. An open subset of a closed subset in this topology is a* quasi-projective variety*.*

A priori, it is not clear whether the Zariski topology is a topology. Further information can be found in [**6**, Proposition 2.1].

As Z provides a map from subsets of a polynomial ring to closed sets in projective space, we now introduce a map which goes in the opposite direction. For any subset V of $\mathbb{P}^n$, we define

$$I(V) := \langle \{f \in K[x_0, \ldots, x_n] \text{ homogeneous} : f(p) = 0 \text{ for all } p \in V\} \rangle.$$

The Nullstellensatz provides a connection between the maps Z, I. They induce a one-to-one correspondence between certain homogeneous ideals in $K[x_0, \ldots, x_n]$ and closed sets in $\mathbb{P}^n$ if K is *algebraically closed*. This means that every nonconstant polynomial in $K[x]$ has a root.

We close the section with the following definition.

Definition 18. *Let $X \subseteq \mathbb{P}^r$ and $X \subseteq \mathbb{P}^s$ be closed subsets, then we define the* product

$$X \times Y := \{[x_0 : \cdots : x_r : y_0 : \cdots : y_s] \in \mathbb{P}^{r+s} : [x_0 : \cdots : x_r] \in X, [y_0 : \cdots : y_s] \in Y\},$$

which is a closed subset of $\mathbb{P}^{r+s+1}$.

2.2. Example: the Grassmannian. We discuss an example of a closed set in projective space, following the reference [**9**, Section 3.2.1].

We can think of $\mathbb{P}^n$ as the set of 1-dimensional vector subspaces of K^{n+1}, by the map

$$[v] \mapsto \operatorname{span}_K\{v\}.$$

It is well defined, because $v \sim v'$ means that there exists a $\lambda \in K \setminus \{0\}$ such that $v = \lambda v'$ and thus $\operatorname{span}_K\{v\} = \operatorname{span}_K\{v'\}$.

We generalize this idea. There exists a closed set in projective space, which parametrizes all d-dimensional subspaces of a vector space K^n. It is called the *Grassmannian*. We briefly explain its construction.

Definition 19. *We define the* Plücker embedding

$$i : \{d\text{-dimensional subspaces of } K^n\} \to \mathbb{P}\left(\bigwedge^d K^n\right)$$

as follows. Let V be a d-dimensional vector subspace of K^n and $v_1, \ldots, v_d$ a basis of V. Then we define its Plücker coordinates *to be*

$$i(V) := v_1 \wedge \cdots \wedge v_d.$$

Lemma 20. *The map i is a well-defined inclusion of sets.*

The following theorem says that the Plücker embedding turns the set of d-dimensional subspaces of K^n into a closed set. We refer to it as the *Grassmannian* and denote it by $Gr(d, K^n)$.

Theorem 21. *The image of the Plücker embedding i is a Zariski-closed subset of $\mathbb{P}(\bigwedge^d K^n)$.*

Finally, we give a concrete description of the Plücler coordinates of a vector space. Recall that the wedge products of unit vectors

$$e_J := \bigwedge_{j \in J} e_j$$

for every subset $J \subseteq [n], |J| = d$ form a basis of $\bigwedge^d K^n$. We can describe the Plücker coordinates of a vector space explicitly in these coordinates.

Lemma 22. *Let $A \in K^{d \times n}$ and let $V \in Gr(d, K^n)$ be the rowspace of A. Then the Plücker coordinate of V is given by*

$$\sum_{\substack{J \subseteq [n] \\ |J| = d}} \det(A_{*,J}) e_J.$$

CHAPTER 2

Polynomial conditions

This chapter is structured as follows. First, we will give a short recapitulation of matroid theory with a focus on matriod duality. In the second section, we use these ideas to show that the set of vector spaces, such that each vector has at least a specific number of nonzero entries, is an open set in the Grassmanian. Finally, we show that there exists an open set, such that the coefficients of the generators of an ideal with certain Hilbert function are in this open set if and only a specific homogeneous degree of that ideal does not contain a polynomial with fewer than a specific number of degrees. This shows that once the Hilbert function of an ideal is known, the question whether it contains a polynomial with less than a specific number of summands is a polynomial condition on its generators.

1. Matroid theory

In this chapter, we briefly recall the basics of matroid theory, following [**10**]. We focus on vector matroids and the connection to their dual matroids, which will become important later.

Definition 23. *Let $\mathcal{M}$ be a pair $(E, \mathcal{B})$, where E is a finite set and $\mathcal{B} \subseteq \mathcal{P}(E)$. Then $\mathcal{M}$ is a matroid, if the following conditions hold:*

(i) $\mathcal{B} \neq \emptyset$
(ii) *If $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 - B_2$ exists a $y \in B_2 - B_1$, such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.*

An element of $\mathcal{B}$ is called a basis of the matroid. If $V \subseteq E$ is a subset of a basis, then we call V independent, otherwise it is called dependent. A circuit of $\mathcal{M}$ is a minimal dependent subset of E.

An important example is the *vector matroid* of a matrix $A \in K^{m \times n}$. We define E to be the set of all column labels of A and $\mathcal{I}$ to be the set of all subsets $X \subseteq E$, such that the columns labelled by X are linearly independent. It is easy to show that this defines a matroid.
Suppose that A has full column rank and let $V \subseteq K^n$ be its rowspace. We can think of V as a point in the Grassmanian $Gr(m, K^n)$, given by its Plücker coordinates. Lemma 22 implies that the set of bases of the vector matroid of A coincides with the set of subsets $B \subseteq [n], |B| = m$, such that the Plücker coordinate of V indexed by B is not zero. Because of this corresondence, we also refer to the vector matroid of A as the vector matroid of the vector space V.

Vector matroids are especially important for finding the inclusion-minimal supports of elements in a vector space. The following definition and proposition, which can be found in [**10**, Proposition 9.2.4], explain this connection.

Definition 24. *Given a matroid $\mathcal{M} = (E, \mathcal{B})$, we define the set $\mathcal{B}^* := \{E - B : B \in \mathcal{B}\}$. Then the pair $\mathcal{M}^* = (E, \mathcal{B}^*)$ is a matroid, to which we refer as the dual of $\mathcal{M}$.*

Proposition 25. *Let $A \in K^{m \times n}$ be a matrix and $\mathcal{M}$ the corresponding vector matroid. Then the set of circuits of the dual matroid $\mathcal{M}^*$ coincides with the set of inclusion-minimal supports of non-zero vectors in the rowspace of A.*

Given a matrix A with full column rank. We can assume that it is of the form $[I_m|D]$, where I_m is the $m \times m$-identity-matrix and $D \in K^{m\times(m-n)}$ arbitrary, by performing elementary row operations and permuting columns. Then the dual matroid is given as the vector matroid of the matrix $[-D^T|I_{m-n}]$, and its circuits are equal to the set of inclusion-minimal supports of non-zero vectors in the rowspace of $[I_m|D]$.
We fix the standard inner product on K^n for the rest of this chapter. Since the orthogonal complement of the rowspace of $[I_m|D]$ is equal to the rowspace of $[-D^T|I_{m-n}]$, we can restate the duality as follows.

Lemma 26. *The set of nonzero vectors with inclusion-minimal support in a vector space V coincides with the set of circuits in the matroid induced by the orthogonal complement $V^\perp$.*

We present a third formulation of that statement. Although it is directly implied by Lemma 26, we give a second proof, which is quite elementary.

Lemma 27. *Let $V \subseteq K^n$ be a K-vector space, $V^\perp$ its orthogonal complement, and $M \in K^{m\times n}$ a matrix with rowspace equals $V^\perp$. Furthermore, let $J \neq \emptyset$ be a subset of $[n]$. Then V contains a nonzero vector v with $\operatorname{supp}(v) \subseteq J$, if and only if $\operatorname{rank} M_{*,J} < |J|$.*

Proof. Let $v \in V \setminus \{0\}$, such that $\operatorname{supp}(v) \subseteq J$. Then we have

$$M_{*,J} v_J = Mv = 0,$$

thus $M_{*,J}$ cannot have full rank.
To proof the converse, suppose that $\operatorname{rank} M_{*,J} < |J|$. It follows that the kernel of $M_{*,J}$ has to be nontrivial. Thus there exists a $v \in K^n$, such that $v_{[n]\setminus J} = 0$ and $M_{*,J} v_J = 0$. But this means $v \perp w$ for all $w \in V^\perp$ and therefore $v \in V$. □

Using the notation of the previous Lemma, a direct consequence is the following.

Lemma 28. *The vector space V contains a nonzero vector with at most t nonzero-entries, if and only if there exists a subset $J \subseteq [n]$ with $|J| = t$, such that for all $I \subseteq [m]$ with $|I| = t$, we have $\det M_{IJ} = 0$.*

2. Polynomial conditions

In this section, we prove that the subset of the Grassmanian consisting of vector spaces such that each vector has at least $t + 1$ nonzero entries is an open set. The key observation for proving this is Lemma 28.

Let us introduce some notation. We refer to a vector space with capital letters, for example V, and to a Plücker coordinate with lower case, indexed by sets $J \subseteq [n]$ with d elements, for example v_J. The collection of Plücker coordinates of a vector space is denoted with brackets, for instance $(v_J)_J$. Furthermore, we define $Gr_J(d, K^n)$ to be the set of all vector spaces V in the Grassmanian $Gr(d, K^n)$, such that v_J is not zero.

We start with the following example.

EXAMPLE 29. *Consider the vector space $V \in Gr(2, K^4)$, which is given as the rowspace of the matrix $\begin{pmatrix} 1 & 0 & a_{11} & a_{12} \\ 0 & 1 & a_{21} & a_{22} \end{pmatrix}$. Then the Plücker coordinates of V are*

$$\begin{array}{rr} v_{\{1,2\}} = 1 & v_{\{1,3\}} = a_{21} \\ v_{\{1,4\}} = a_{22} & v_{\{2,3\}} = -a_{11} \\ v_{\{2,4\}} = -a_{12} & v_{\{3,4\}} = a_{11}a_{22} - a_{21}a_{12} \end{array}.$$

Then the rowspace of the matrix $\begin{pmatrix} -a_{11} & -a_{21} & 1 & 0 \\ -a_{12} & -a_{22} & 0 & 1 \end{pmatrix}$ is the orthogonal complement $V^\perp$ of V. We denote the Plücker coordinates of $V^\perp$ by $v_J^\perp$ and calculate

$$\begin{array}{rr} v^\perp_{\{1,2\}} = a_{11}a_{22} - a_{21}a_{12} & v^\perp_{\{1,3\}} = a_{12} \\ v^\perp_{\{1,4\}} = -a_{11} & v^\perp_{\{2,3\}} = a_{22}. \\ v^\perp_{\{2,4\}} = -a_{21} & v^\perp_{\{3,4\}} = 1 \end{array}$$

We observe that we can calculate a basis of a vector space from its Plücker coordinates. The next definition and the next lemma make this precise.

DEFINITION 30. *Let $\tilde{J} = \{\tilde{j}_1, \ldots, \tilde{j}_d\} \subseteq [n]$, such that $\tilde{j}_i < \tilde{j}_{i+1}$. Moreover, we define bijective maps $\sigma_{ij} : [d] \to (\tilde{J} \cup \{j\}) \setminus \{\tilde{j}_i\}$ for all $i \in [d], j \in [n] \setminus \tilde{J}$, such that*

$$\sigma_{ij}(l) := \begin{cases} \tilde{j}_l, & l \neq i \\ j, & \text{otherwise} \end{cases}.$$

We define a polynomial map $\phi_{\tilde{J}} : Gr_{\tilde{J}}(d, K^n) \to \mathbb{P}(K^{d\times n})$, such that for all $V \in Gr_{\tilde{J}}(d, K^n)$

$$\phi_{\tilde{J}}(V)_{*,\tilde{J}} := v_{\tilde{J}} I_d,$$

where I_d denotes the $d \times d$ identity matrix, and

$$(\phi_{\tilde{J}}(V))_{ij} := \operatorname{sgn}(\sigma_{ij}) v_{(\tilde{J}\cup\{j\})\setminus\{\tilde{j}_i\}} \text{ for all } i \in [d] \text{ and for all } j \in [n] \setminus \tilde{J}.$$

Note that the map $\phi_{\tilde{J}}$ is well-defined. Given Plücker coordinates $(v_J)_J, (w_J)_J$ of V, with $v_J = \lambda w_J$, for a $\lambda \in K \setminus \{0\}$ and for all $J \subseteq [n], |J| = d$. We calculate

$$v_{\tilde{J}} I_d = \lambda(w_{\tilde{J}} I_d),$$

and

$$\operatorname{sgn}(\sigma_{ij}) v_{(\tilde{J}\cup\{j\})\setminus\{\tilde{j}_i\}} = \lambda(\operatorname{sgn}(\sigma_{ij}) w_{(\tilde{J}\cup\{j\})\setminus\{\tilde{j}_i\}}),$$

for all $i \in [d]$ and for all $j \in [n] \setminus \tilde{J}$. It follows that $\phi_{\tilde{J}}((v_J)_J) = \lambda\phi_{\tilde{J}}((w_J)_J)$.

LEMMA 31. *Given a vector space $V \in Gr(d, K^n)_{\tilde{J}}$. Then the rowspace of $\phi_{\tilde{J}}(V)$ is equal to V.*

Proof. Fix Plücker coordinates $(v_J)_J$ for V. Since $v_{\tilde{J}} \neq 0$, there exists a matrix $M = (m_{ij}) \in K^{d\times n}$, such that the rowspace of M is equal to V and $M_{*,\tilde{J}} = I_d$.

Since $\det M_{*,\tilde{J}} = 1$, it follows that $\det M_{*,J} = v_{\tilde{J}}^{-1} v_J$ for all $J \subseteq [n], |J| = d$. Moreover,

$$\begin{aligned}
(\phi_{\tilde{J}}(V))_{ij} &= \operatorname{sgn}(\sigma_{ij}) v_{(\tilde{J}\cup\{j\})\setminus\{\tilde{j}_i\}} \\
&= \operatorname{sgn}(\sigma_{ij}) v_{\tilde{J}} \det M_{(\tilde{J}\cup\{j\})\setminus\{\tilde{j}_i\}} \\
&= \operatorname{sgn}(\sigma_{ij}) v_{\tilde{J}} \sum_{\substack{\sigma:[d]\to(\tilde{J}\cup\{j\})\setminus\{\tilde{j}_i\} \\ \sigma \text{ bijective}}} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} m_{i,\sigma(i)} \\
&= \operatorname{sgn}(\sigma_{ij}) v_{\tilde{J}} \operatorname{sgn}(\sigma_{ij}) m_{ij} \\
&= v_{\tilde{J}} m_{ij},
\end{aligned}$$

for all $i \in [d]$ and for all $j \in [n] \setminus \tilde{J}$. Since $\phi_{\tilde{J}}(V)_{*,\tilde{J}} = v_{\tilde{J}} I_d = v_{\tilde{J}} M_{*,\tilde{J}}$, we have $\phi_{\tilde{J}}(V) = v_{\tilde{J}} M$. This implies that the rowspace of $\phi_{\tilde{J}}(V)$ is equal to V. □

We also observe from Example 29 that we get the Plücker coordinates of the orthogonal complement directly from the Plücker coordinates of the vector space.

Definition 32. *We define the map*

$$\psi_J : Gr_J(d, K^n) \to Gr_{[n]\setminus J}(n-d, K^n),$$

which sends a vector space to its orthogonal complement.

Lemma 33. *The map ψ_J is polynomial.*

Proof. Let $J = \{j_1, \dots, j_d\}$, such that $j_i < j_{i+1}$. We define two maps as follows. First, let

$$\begin{aligned}
\varphi : \mathbb{P}(K^{d\times n})_J &\to \mathbb{P}(K^{(n-d)\times n})_{[n]\setminus J} \\
\varphi(M)_{*,[n]\setminus J} &:= m_{1,j_1} I_{n-d} \\
\varphi(M)_{*,J} &:= -(M_{*,[n]\setminus J})^T,
\end{aligned}$$

where $\mathbb{P}(K^{d\times n})_J := \{M \in \mathbb{P}(K^{d\times n}) | M_{*,J} = \lambda I_d \text{ for a } \lambda \in k \setminus \{0\}\}$ and $M = (m_{ij})_{ij}$. Second, we define

$$\begin{aligned}
\tilde{\psi} : Gr_J(d, K^n) &\to Gr_{[n]\setminus J}(n-d, K^n) \\
(v_J)_J &\mapsto (det(\varphi(\phi_J(V))_{*,J}))_J.
\end{aligned}$$

Note that $\tilde{\psi}$ is well defined and polynomial, since the determinant is a homogeneous polynomial.
Since the rowspace of $\phi_J(V)$ is equal to V, it follows that the rowspace of $\varphi(\phi_J(V))$ is equal to the orthogonal complement of V. Thus we have $\psi_J = \tilde{\psi}$, which finishes the proof. □

We are now able to prove the following proposition.

Proposition 34. *Let $t \le n-d$ and $\mathcal{V}_t$ be the set of all d-dimensional subspaces V of K^n, such that each nonzero vector in V has at least $t+1$ nonzero entries. Then $\mathcal{V}_t$ is an open set in the Grassmanian $Gr(d, K^n)$.*

Proof. Using Lemma 28, it is clear that $\mathcal{V}_t \cap Gr(d, K^n)_{\tilde{J}}$ is equal to

$$\{V \in Gr(d, K^n)_{\tilde{J}} : \text{for all } J \subseteq [n], |J| = t, \text{ there exist an } I \subseteq [n-d], |I| = t,$$
$$\text{such that } \det((\phi_{[n]\setminus\tilde{J}}(\psi_{\tilde{J}}(V)))_{I,J}) \neq 0\}.$$
$$= \bigcap_{\substack{J\subseteq[n]\\|J|=t}} \bigcup_{\substack{I\subseteq[n-d]\\|I|=t}} Gr(d, K^n)_{\tilde{J}} \setminus Z(\det((\phi_{[n]\setminus\tilde{J}}(\psi_{\tilde{J}}(V)))_{I,J})),$$

which is an open set, because $\phi_{\tilde{J}}$ and $\psi_{\tilde{J}}$ are polynomial maps. Thus

$$\mathcal{V}_t = \bigcup_{\substack{\tilde{J}\subseteq[n]\\|\tilde{J}|=d}} \mathcal{V}_t \cap Gr(d, K^n)_{\tilde{J}}$$

is open too. □

3. Polynomial conditions on the coefficients of the generators

The goal of this chapter is to apply the proposition from the last section to our case of polynomial ideals. We show that there exists a Zariski open set, such that the coefficients of the generators of an ideal with certain Hilbert function are in this open set if and only a specific homogeneous degree of that ideal does not contain a polynomial with fewer than $t+1$ summands.

We start by defining a moduli space for homogeneous ideals. Since we want to use algebraic geometry, we restrict the degrees of the generating polynomials.

DEFINITION 35. *Fix a natural number D and sets $\mathcal{M}_d := \{\alpha \in \mathbb{N}_0^{n+1} : |\alpha| = d\}$ for all $d = 0, \dots D$. We define the variety*

$$\mathcal{C}_D := \prod_{d=0}^{D} \prod_{E\subseteq\mathcal{M}_d} \mathbb{P}K^E.$$

We denote a coordinate of a point $p \in \mathcal{C}_{D,n}$ by $p_{E,e}$, where $d = 0, \dots, D$, $E \subseteq \mathcal{M}_d$ and $e \in E$.
Moreover, we define the map

$$\mathcal{I}_D : \mathcal{C}_D \to \{\textit{homogeneous ideals } I \subseteq K[x_0, \dots, x_n] \textit{ with generators of degree at most } D\}$$
$$p \mapsto \langle \sum_{e\in E} p_{E,e} x^e : E \subseteq \mathcal{M}_d \textit{ for all } d = 0, \dots, D\rangle.$$

The following lemma enables us to think of $\mathcal{C}_D$ as the collection of all homogeneous ideals in $K[x_0, \dots, x_n]$, such that there exist generators of degree at most D.

LEMMA 36. *The map $\mathcal{I}_D$ is surjective.*

Proof. Given an ideal $I = \langle f_1, \dots, f_n\rangle$, where the generators are homogeneous and their supports are pairwise distinct. This ideal is in the image of $\mathcal{I}_D$. Now suppose that f_1 and f_2 have the same support. Then we can find a linear combination $f := \lambda_1 f_1 + \lambda_2 f_2 \in I, \lambda_1, \lambda_2 \in K$, such that $\text{supp}(f) \subsetneq \text{supp}(f_1)$. Moreover, we have $I = \langle f, f_2, \dots, f_n\rangle$. Repeating this step completes the proof. □

Given generators of a polynomial ideal, we get a generating set for an arbitrary homogeneous part of that ideal by multiplying every suitable generator with a monomial of a certain nonzero degree. The following definition and lemma make this precise.

Definition 37. *Let $p \in \mathcal{C}_D$ be a point with corresponding ideal $I := \mathcal{I}_D(p)$ and $\tilde{d}$ a natural number. We define the matrix $M^{(\tilde{d})}(p)$, such that its rows are the coordinates of the polynomials*

$$x^\alpha \sum_{e \in E} p_{E,e} x^e = \sum_{e \in E} p_{E,e} x^{\alpha+e}$$

for every $E \subseteq \mathcal{M}_d, d \le \min(\tilde{d}, D)$, and $\alpha \in \mathcal{M}_{\tilde{d}-d}$, in the monomial basis $\{x^\beta : \beta \in \mathcal{M}_{\tilde{d}}\}$.

Lemma 38. *The rowspace of $M^{(\tilde{d})}(p)$ is equal to the $\tilde{d}$-th homogeneous component $I^{(\tilde{d})}$ of I.*

Proof. We have to show that elements of the form $\sum_{e \in E} p_{E,e} x^{\alpha+e}$ generate $I^{(\tilde{d})}$. Let f be an arbitrary polynomial in $I^{(\tilde{d})}$. Thus we can write

$$f = \sum_{d=0}^{\tilde{d}} \sum_{E \subseteq \mathcal{M}_d} g_E \sum_{e \in E} p_{E,e} x^e,$$

with polynomials $g_E = \sum_{\alpha \in \mathcal{M}_{\tilde{d}-d}} g_{E,\alpha} x^\alpha$, $E \subseteq \mathcal{M}_d$. A calculation shows

$$\begin{aligned} f &= \sum_{d=0}^{\tilde{d}} \sum_{E \subseteq \mathcal{M}_d} g_E \sum_{e \in E} p_{E,e} x^e \\ &= \sum_{d=0}^{\tilde{d}} \sum_{E \subseteq \mathcal{M}_d} \sum_{\alpha \in \mathcal{M}_{\tilde{d}-d}} g_{E,\alpha} x^\alpha \sum_{e \in E} p_{E,e} x^e \\ &= \sum_{d=0}^{\tilde{d}} \sum_{E \subseteq \mathcal{M}_d} \sum_{\alpha \in \mathcal{M}_{\tilde{d}-d}} g_{E,\alpha} \sum_{e \in E} p_{E,e} x^{\alpha+e}, \end{aligned}$$

as required. □

Next, we introduce the following subset of $\mathcal{C}_D$.

Definition 39. *Let $H : \mathbb{N}_0 \to \mathbb{N}_0$ be a map and let $t, \tilde{d}, D$ be natural numbers, such that $t \le |\mathcal{M}_{\tilde{d}}| - H(\tilde{d})$. Then we define the set $\mathcal{V}_{H,t}^{(\tilde{d})} \subseteq \mathcal{C}_D$ as*

$$\begin{aligned} \mathcal{V}_{H,t}^{(\tilde{d})} := \{ & p \in \mathcal{C}_D : \textit{there exist } \tilde{I}, \tilde{J}, |\tilde{I}| = |\tilde{J}| = H(\tilde{d}) \textit{ such that } \det M^{(\tilde{d})}(p)_{\tilde{I},\tilde{J}} \neq 0 \\ & \textit{and the minors of } M^{(\tilde{d})}(p) \textit{ with row label } \tilde{I} \textit{ are in } \mathcal{V}_t \subseteq Gr(H(\tilde{d}), k^{|\mathcal{M}_{\tilde{d}}|})\}. \end{aligned}$$

Clearly, $\mathcal{V}_{H,t}^{(\tilde{d})}$ is open, because the entries of $M^{(\tilde{d})}(p)$ consists only of zeros and coordinates of p, and $\mathcal{V}_t$ is open.

This set gives a description for homogeneous ideals, in which the minimal number of summands of a polynomial is bounded from below, which can be seen with Proposition 34. This is also the statement of the next theorem.

THEOREM 40. *Let $H : \mathbb{N}_0 \to \mathbb{N}_0$ be a map and let $t, \tilde{d}, D$ be natural numbers, such that $t \leq |\mathcal{M}_{\tilde{d}}| - H(\tilde{d})$. Then we have*

$$\{I \in \mathcal{I}_D(\mathcal{V}_{H,t}^{(\tilde{d})}) : I \text{ has Hilbert function } H\} = \left\{ \begin{array}{l} \text{homogeneous ideals } I \subseteq K[x_0, \dots, x_n] \text{ with} \\ \text{Hilbert function } H \text{ and generators of degree} \\ \text{at most } D \text{ such that every nonzero polynomial} \\ \text{in } I^{(\tilde{d})} \text{ has at least } t+1 \text{ summands} \end{array} \right\}$$

Proof. Let $p \in \mathcal{V}_{H,t}^{(\tilde{d})}$, such that $I := \mathcal{I}_D(p)$ has Hilbert function H. Then there exist $\tilde{I}, \tilde{J}, |\tilde{I}| = |\tilde{J}| = H(\tilde{d})$ such that $\det M^{(\tilde{d})}(p)_{\tilde{I},\tilde{J}} \neq 0$. But since the rowspace of $M^{(\tilde{d})}(p)$ is $I^{(\tilde{d})}$ and $\dim_k I^{(\tilde{d})} = H(\tilde{d})$, it follows that the rowspace of $M^{(\tilde{d})}(p)_{\tilde{I},*}$ is equal to $I^{(\tilde{d})}$. Because this restricted matrix has full rank, its maximal minors are Plücker coordinates of $I^{(\tilde{d})}$. Since they are in $\mathcal{V}_t$, there cannot exist a polynomial with fewer than $t+1$ summands in $I^{(\tilde{d})}$, by Proposition 34.
To proof the converse, given an homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$ with Hilbert function H and generators of degree at most D, such that every nonzero polynomial in $I^{(\tilde{d})}$ has at least $t+1$ summands. Because of Lemma 36, there exists a $p \in \mathcal{C}_D$, such that $I = \mathcal{I}_D(p)$. Furthermore, since the rowspace of $M^{(\tilde{d})}(p)$ is $I^{(\tilde{d})}$ and $\dim_k I^{(\tilde{d})} = H(\tilde{d})$, there exist $\tilde{I}, \tilde{J}, |\tilde{I}| = |\tilde{J}| = H(\tilde{d})$ such that $\det M^{(\tilde{d})}(p)_{\tilde{I},\tilde{J}} \neq 0$. This implies that the minors of $M^{(\tilde{d})}(p)$ with row label $\tilde{I}$ are Plücker coordinates of $I^{(\tilde{d})}$. By assumption, they are in $\mathcal{V}_{H,t}^{(\tilde{d})}$. □

This theorem leads to the question, whether the *Hilbert scheme* might be a more natural moduli space than $\mathcal{C}_D$. The Hilbert scheme parametrizes closed subschemes of projective space. There is a theorem in Algebraic Geometry, which says that a closed subscheme of projective space corresponds to a saturated homogeneous ideal in the polynomial ring (see for example [**11**, Exercise III-16]). Thus the Hilbert scheme parametrizes saturated homogeneous ideals of a polynomial ring. But since a homogeneous ideal contains a polynomial with fewer than t summands if and only if its saturation does, this is not serious restriction. Another advantage of the Hilbert scheme is that it is a moduli space for saturated homogeneous ideals with fixed Hilbert function. We had to assume this additionally in the theorem.

CHAPTER 3

Determinantal ideals

The determinant is a central object in linear algebra, but it is also interesting in other branches of mathematics. In this chapter, we present the determinant with a view toward Combinatorics and Commutative Algebra. Our introduction includes basic definitions, a discussion of computitional aspects and a small selection of relatively young results, most notably the straightening formula.

1. Determinants

First, we collect basic material about the determinant. Our main reference is [**12**, Chapter 4].

Definition 41. *Let V be an n-dimensional K-vector space. A* determinant function *on V is an alternating multilinear map $\Delta : V^n \to K$. It is called* nontrivial *if it is not the zero map.*

The following theorem can be found in [**12**, Section 4.2., Korollar 9].

Theorem 42. *Let Δ, Δ' be determinant functions on the K-vector space V. If Δ is nontrivial, there exists a unique $\lambda \in K$ such that $\Delta' = \lambda \cdot \Delta$.*

If we fix a basis $V \cong K^n$, we get an isomorphism $V^n \cong K^{n\times n}$. Thus we can think of a determinant function as a map

$$\Delta : K^{n\times n} \to K,$$

which is alternating und multilinear in the coloumns of matrices in $K^{n\times n}$. Then Theorem 42 states that a nontrivial determinant function on K^n is unique up to scaling. In particular, if we know one nontrivial determinant function on K^n, we know every determinant function on K^n.
The next natural step would be to ask, whether a nontrivial determinant function exists. The answer is yes. The *Leibniz-formula* concretly describes for every $n \in \mathbb{N}$ a determinant function det on $K^{n\times n}$ satisfying $\det(I) = 1$, where I denotes the identity matrix $I \in K^{n\times n}$. Every determinant function on K^n is of the form $\lambda \cdot \det$, where $\lambda \in K$.
We establish notions necessary for understanding the Leibniz-formula in the remainder of this section.

Definition 43. *We define the* symmetric group *S_n for $n \geq 2$. As a set, S_n consists of all bijective maps $[n] \to [n]$. The group operation is given by composition. Moreover, the elements of S_n are called* permutations*. A permutation $\sigma \in S_n$ is called* transposition*, if it swaps two distinct $k, l \in [n]$ and fixes every other element.*

It is easy to prove that the symmetric group has cardinality $S_n = n!$.

Definition 44. *The* signum *of a permutation* $\sigma \in S_n$ *is*

$$\operatorname{sgn}(\sigma) := \prod_{1\leq i<j\leq n} \frac{\sigma(j)-\sigma(i)}{j-i}.$$

Let us calculate an example. The identity $id \in S_n$ has signum

$$\operatorname{sgn}(id) = \prod_{1\leq i<j\leq n} \frac{id(j)-id(i)}{j-i} = \prod_{1\leq i<j\leq n} \frac{j-i}{j-i} = 1.$$

Also, if $\tau \in S_n$ is a transposition we can calculate $\operatorname{sgn}(\tau) = -1$. The following proposition generalizes this observation.

Proposition 45. *If a permutation* $\sigma \in S_n$ *is a product of* s *transpositions, we have* $\operatorname{sgn}(\sigma) = (-1)^s$. *Moreover,* sgn *is a surjective group homomorphism from the symmetric group onto* $(\{-1,1\},\cdot)$.

Since sgn is a homomorphism, its kernel is a normal subgroup of S_n.

Definition 46. *The* alternating group A_n *is the kernel of* sgn*, in particular*

$$A_n := \{\sigma \in S_n \colon \operatorname{sgn}(\sigma) = 1\}.$$

Lemma 47. *The alternating group has cardinality* $|A_n| = \frac{n!}{2}$.

Proof. The surjectivity of sgn implies

$$S_n/A_n \cong (\{1,-1\},\cdot).$$

In particular, the index of A_n in S_n is 2. Using *Lagrange's Theorem* (for example, [**13**, Theorem 8]), we get

$$|A_n| = \frac{|S_n|}{[S_n : A_n]} = \frac{n!}{2}.$$

□

We are now able to write down a nontrivial determinant function of an n-dimensional vector space explicitly. This proves the existence of a nontrivial determinant function on every finite-dimensional vector space, which is unique up to scaling. Moreover, this enables use to speak of the *determinant*, which will always mean the map det in the following proposition.

Proposition 48 (Leibniz formula). *The function*

$$\begin{aligned} \det\colon K^{n\times n} &\to K \\ A = (a_{ij})_{ij} &\mapsto \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \end{aligned}$$

is a nontrivial determinant function on K^n *satisfying* $\det(I) = 1$*, where* $I \in K^{n\times n}$ *denotes the identity matrix.*

Proof. We just have to check all properties of the determinant, which are straightforward calculations. See [**12**, Section 4.2, Lemma 4], for instance. □

2. Interlude: Computational aspects of the determinant and its cousin

In this section, we give some comments on the calculation of the determinant. But why should somebody be interested in it?

THEOREM 49. *A matrix $A \in K^{n\times n}$ is invertible if and only if* $\det(A) \neq 0$.

Proof. [**12**, Section 4.3, Satz 4, (iv)] ☐

One can generalize this theorem to lower ranks of an even non-quadratic matrix.

DEFINITION 50. *Let $A \in K^{m\times n}$ and $0 < t \leq \min\{m, n\}$. A t-*minor *of A is the determinant of a $t \times t$-submatrix of A.*

THEOREM 51. *Every t-minor of a matrix vanishes if and only if the matrix has rank at most $t-1$.*

Proof. [**12**, Section 4.3, Korollar 5, (iv)] ☐

The rank of a matrix is of fundamental importance in applications. For example, consider a system of linear equation $Ax = b$, where $A \in K^{m\times n}, x \in K^n$ and $b \in K^m$. The following facts, which are proven in [**12**, Section 3.5, Satz 5], reveal the relationship between the rank of A and the solvability of $Ax = b$.

- $Ax = b$ is solvable if and only if $\operatorname{rank} A = \operatorname{rank}(A|b)$.
- $Ax = b$ is solvable for every $b \in K^m$ if $\operatorname{rank} A = m$
- $Ax = b$ has at most one solution for every $b \in K^m$ if $\operatorname{rank} A = n$.

This points out, why one should be interested in calculating determinants of matrices.

How to calculate the determinant of an $n\times n$-matrix? The Leibniz-formula gives a first method. Unfortunately, it has $n!$ summands, so the number of arithmetic operations needed is at least $O(n!)$. This is not an efficient algorithm. Surprisingly, one can give a much better method. Consider the following properties of the determinant.

THEOREM 52. *Let $a \in K, A \in K^{n\times n}$ and $i, j \in [n]$.*

(i) $\det(A)$ *does not change, if adding to the i-th row (resp. column) a-times of the j-th row (resp. column).*
(ii) $\det(A)$ *changes the sign, if swapping the i-th row (resp.column) and the j-th row (resp. column)*
(iii) *multiplying* $\det(A)$ *with a is the same as multiplying a row (resp. column) of A with a.*

Proof. [**12**, Section 4.3,Satz 6] ☐

The operations (i), (ii), and (iii) with $a \neq 0$ are called *elementary row* (resp. *column*) *operations*.

Recall that the *Gaussian elimination* produces out of a matrix $A \in K^{n\times n}$ a matrix $\tilde{A} = (\tilde{a}_{ij})_{ij} \in K^{n\times n}$ such that $\tilde{a}_{ij} = 0$ for all $i > j$, by elementary row operations. But the theorem above garanties that $\det(A)$ and $\det(\tilde{A})$ only differ by a sign. Also, it is easy to calculate the determinant of $\tilde{A}$.

LEMMA 53. *Let $\tilde{A} \in K^{n\times n}$ such that $\tilde{a}_{ij} = 0$ for all $i > j$. Then we have*

$$\det(\tilde{A}) = \tilde{a}_{11} \cdots \tilde{a}_{nn}.$$

Proof. This can be seen directly with the Leibniz formula. ☐

To summary, we can think of the Gaussian elemination as an algorithm for computing the determinant of a matrix. It turns out that this algorithm is much faster than the Leibniz-formula.

THEOREM 54. *Gaussian elimination of a quadratic matrix $A \in K^{n\times n}$ can be performed using at most $O(n^3)$ arithmetic operations.*

Proof. Gaussian elimination has m iterations, one for each row of A. During one iteration, the algorithm performs at most n elementary row operations. Since a row operation consists of at most $O(n)$ arithmetic operations, we conclude that Gaussian elimination can be performed using $O(n^3)$ arithmetic operations. □

There are even ways to improve the complexity of this algorithm in certain special cases. Also, one can improve the performance of this algorithm by using matrix decompositions. But because this bachelor thesis is primarily concerend with the algebra and combinatorics of determinants, we will not go deeper into it.

For the rest of this section, we discuss the following question. Why is it possible to reduce the computitional complexity of the Leibniz-formula? An answer is: because of the signum homomorphism. To illustrate this, let us copy the Leibniz-formula but forget about sgn in it.

DEFINITION 55. *The* permanent *of a matrix $A = (a_{ij})_{ij} \in K^{n\times n}$ is*

$$\operatorname{per}(A) := \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Permanents occur naturally in combinatorial problems. As an example, choose natural numbers $m \leq n$ and subsets $X_i \subseteq [n]$ where $i = 1, \dots, n$. A sequence of elements $s_1, \dots, s_n \in [n]$ is called *system of distinct representatives* if they are pairwise distinct and $s_i \in X_i$. Now define a matrix $A = (a_{ij})_{ij} \in \{0,1\}^{m\times n}$ such that $a_{ij} = 1$ if and only if $j \in X_i$. Then clearly $\prod_{i=1}^{m} a_{i,\sigma(i)} = 1$ if $\sigma(1), \dots, \sigma(m)$ is a system of distinct representatives. Thus the number of systems of distinct representatives is equal to the following generalisation of the permanent

$$\sum_{\substack{\text{injections}\\ [m]\to[n]}} \prod_{i=1}^{m} a_{i,\sigma(i)}.$$

But how to calculate the permanent of a matrix? It turns out that the permanent is not well-behaved under row operations. For example, the permanent of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{Q}^{2\times 2}$ is

$$\operatorname{per}(A) = 2,$$

but subtracting the first from the second row yields

$$\operatorname{per}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = 0.$$

In particular, we cannot use Gaussian elimination to calculate the permanent of a matrix. But things are much more worse than that. Even for matrices with entries in 0 and 1, we have the following result.

THEOREM 56. *The complexity of computing the permanent of a matrix $A \in \{0,1\}^{n\times n}$ is NP-hard.*

Proof. [**14**, Theorem 1] □

Because of the difficulties in computing permanents, a great deal of research was put into finding good upper and lower bounds. Further information can be found in [**15**], which was our reference in the last section.

3. Determinants in Combinatorics and Commutative Algebra

Let us extend the definition of the determinant to polynomial rings.

DEFINITION 57. *Let $A = (a_{ij})_{ij}$ be an $n \times n$-matrix with entries in a polynomial ring. Then its* determinant *is*

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

A t-minor of A is the determinant of a $t \times t$-submatrix.

Generally speaking, the determinant is interesting for algebraists because of two reasons. First, it is a polynomial. Second, because of its combinatorial properties. We sketch some of these ideas in the following two passages.

3.1. The determinant as a polynomial. Let $X = (x_{ij})_{ij}$ be a matrix of indeterminants over K and $I \subseteq [m], J \subseteq [n], |I| = |J| = t$. Then $\det(X)$ is an homogeneous polynomial of degree t. Moreover, the minor $\det(A_{I,J})$ is

$$\det(A_{I,J}) = (\det(X_{I,J}))(A),$$

for every $A \in K^{m \times n}$. This observation leads to the following theorem.

THEOREM 58. *The set of matrices $A \in K^{m\times n}$ of rank $\leq t-1$ is a closed set in projective space $\mathbb{P}(K^{m\times n})$ given by the vanishing of all t-minors of X.*

This closed set is called *determinantal variety* and is a classical example in projective geometry.
Surprisingly, one can show, that the t-minors are the only polynomials which detect the rank of a matrix. This is made precise by saying that the ideal of a determiantal variety is given by the following

DEFINITION 59. *For $0 < t \leq \min\{m, n\}$, the* determinatal ideal *I_t in $K[X]$ is*

$$I_t := \langle t\text{-minors of } X \rangle$$

We apply concepts from the first chapter to this special class of ideals.

THEOREM 60. *Determinantal ideals are prime.*

Proof. [**16**, Corollary 16.29] □

COROLLARY 61. *If $t \geq 2$, the determinantal ideal I_t does not contain a monomial.*

Proof. Suppose that I_t contains a monomial $x_{i_1,j_1} \cdots x_{i_k,j_k}$. Since I_t is prime, it follows that there exist a $l \in [k]$ such that $x_{i_l,j_l} \in I_t$. But the generators of I_t are homogeneous of degree $t \geq 2$, so the determinantal ideal cannot contain a single variable. □

THEOREM 62. *The t-minors of X form a reduced Gröbner basis of I_t with respect to the lexicographic term order induced from the variable order*

$$x_{1,n} > \cdots > x_{1,1} > x_{2,n} > \cdots > x_{2,1} > \cdots > x_{m,n} > \cdots > x_{m,1}.$$

Proof. [**17**, Theorem 1]. □

COROLLARY 63. *Let $t \geq 2$, m a t-minor of X and $x^\alpha \in K[X]$ a monomial. Then* $\operatorname{supp}(x^\alpha m)$ *is a circuit of I_t.*

Proof. By Lemma 9 and Theorem 62, we conclude that $\operatorname{supp}(m)$ is a circuit of I_t. Now choose a polynomial $f \in I_t$ such that $\operatorname{supp}(f) \subseteq \operatorname{supp}(x^\alpha m)$. It follows that we can write $f = x^\alpha f'$, where $\operatorname{supp}(f') \subseteq \operatorname{supp}(m)$. Since I_t is prime and $x^\alpha \notin I_t$ by Corollary 61, we conclude $f' \in I_t$. But this implies $\operatorname{supp}(f') = \operatorname{supp}(m)$, since $\operatorname{supp}(m)$ is a circuit of I_t. Also, it follows that $\operatorname{supp}(f) = \operatorname{supp}(x^\alpha m)$, which proves that $\operatorname{supp}(x^\alpha m)$ is a circuit of I_t. □

Unfortunately, Lemma 9 does not provide a full characterisation of circuits of I_t; they remain mysterious. We return back to this problem in the last chapter, where we consider a related problem: What is the minimal number of summands of a nonzero polynomial in I_t?

3.2. The combinatorics of determinants. The proof for Theorem 62 in [**17**] relies on the straightening formula. We explain this result following [**18**], to give an example of a connection between the combinatorics of determinants and Commutative Algebra. As a reward, we will be able to describe the homogeneous components of determinantal ideals explicitly.

A *diagram* is a finite subset $\sigma \subseteq \mathbb{N}^2$, such that if $(i,j) \in \sigma$ and $\tilde{i} \leq i, \tilde{j} \leq j$, then $(\tilde{i},\tilde{j}) \in \sigma$. A diagram σ is determined by its sequence of *row lengths* $\sigma_i = \max_{(i,j)\in\sigma} j$. We define the *Snapper order* on the set of diagrams: $\sigma \geq \tau$ if $\sum_{i=1}^k \sigma_i \geq \sum_{i=1}^k \tau_i$ for all k.
A *tableau T of shape* σ is a map $T : \sigma \to [n]$. We use the notation $|T| = \sigma$ and think of T as a way of filling in the "boxes of σ" with numbers between 1 and n. The *content* of a tableau T is the map

$$\begin{aligned} C_T\colon [n] &\to \mathbb{N}_0 \\ p &\mapsto \text{number of times } p \text{ occurs in } T. \end{aligned}$$

We define an ordering on the set of tableaux as follows. We say $S \leq T$ if for all p, q the first p rows of S contain fewer occurrences of integers at most q than the corresponding rows of T. One can prove that $S \leq T$ implies that $|S| \leq |T|$.
A *double tableau* is a pair of two tableaux $(S|T)$, such that both S and T have the same shape. We can transfer the notions for tableaux to this setting. If $S \leq \tilde{S}$ and $T \leq \tilde{T}$, we say that $(S|T) \leq (\tilde{S}|\tilde{T})$. The *content* of $(S|T)$ is the pair $C_{(S|T)} = (C_S, C_T)$. We think of double tabelaux as products of minors, associating to $(S|T)$ the product of minors of X whose i-th factor is the minor involving rows $S(i,1), S(i,2), \ldots$ and columns $T(i,1), T(i,2), \ldots$.
A tableau is *standard*, if its rows are strictly increasing sequences and its columns are non-decreasing sequences. A double tableau $(S|T)$ is *standard*, if S and T are standard.

We are now able to state the theorem. In particular, the straightening formula establishes a connection between the combinatorics of double standard tableaux and Commutative Algebra.

THEOREM 64 (Doubilet, Rota, Stein). *The double standard tableaux form a K-basis for the polynomial ring $K[X]$. Moreover, if*

$$M = \sum_{i=1}^{n} \lambda_i M_i$$

with $\lambda \in K \setminus \{0\}$ for $i = 1, \ldots, n$ is the unique expression of a double tableau M as a linear combination of distinct double standard tableaux then $M_i \geq M$ and $C_{M_i} = C_M$ for each $i = 1, \ldots, n$.

As an application, we describe the homogeneous components of determinantal ideals explicitly. This also gives a description of the Hilbert function of I_t.

COROLLARY 65. *Let $B_t^{(d)}$ be the set of double standard tableaux $(S|T)$ such that $\sigma = |S| = |T|$ satisfies $\sigma_1 \geq t$ and $\sum_i \sigma_i = d$. Then $B_t^{(d)}$ is a K-basis of $I_t^{(d)}$.*

Proof. Because of Theorem 64, it is clear that $B_t^{(d)}$ is linearly dependent. It remains to show that it spans the homogeneous component.
First, we prove that $\operatorname{span} B_t^{(d)} \subseteq I_t^{(d)}$. Let $M = (S|T) \in B_t^{(d)}$, with shape $\sigma = |S| = |T|$. It follows that $M \in I_{\sigma_1}$. We also have $\sigma_1 \geq t$ by the definition of $B_t^{(d)}$, so the *Laplace expansion* implies $I_{\sigma_1} \subseteq I_t$ and thus $M \in I_t$.
Next, we show that $I_t^{(d)} \subseteq \operatorname{span} B_t^{(d)}$. We denote the generators of I_t by $m_{I,J} = \det(X_{I,J})$. Then a polynomial $f \in I_t^{(d)}$ has the form $f = \sum_{I,J} f_{I,J} m_{I,J}$, where the sum varies over all generators and $f_{I,J}$ is an homogeneous polynomial of degree $d-t$. By Theorem 64, we can write $f_{I,J}$ as a linear combination of double standard tableaux

$$f_{I,J} = \lambda_{I,J} \sum_l M_{I,J}^{(l)},$$

where $\lambda_{I,J} \in K$. Since $M_{I,J}^{(l)}$ is homogeneous, we assume that $\deg M_{I,J}^{(l)} = d - t$. In particular, $|M_{I,J}^{(l)}| = \sigma_{I,J}^{(l)}$ satisfies $\sum_i (\sigma_{I,J}^{(l)})_i = d - t$.
We define the double tableaux $\tilde{M}_{I,J}^{(l)} := M_{I,J}^{(l)} m_{I,J}$. Its shape $|\tilde{M}_{I,J}^{(l)}| = \tilde{\sigma}_{I,J}^{(l)}$ satisfies $(\tilde{\sigma}_{I,J}^{(l)})_1 \geq t$ and $(\sum_i \tilde{\sigma}_{I,J}^{(l)})_i = (d-t) + t = d$. Using the straightening law, we write these tableaux as linear combinations

$$\tilde{M}_{I,J}^{(l)} = \sum_s \mu_s \tilde{M}_{I,J,s}^{(l)}$$

with $\mu_s \in K$ of double standard tableaux $\tilde{M}_{I,J,s}^{(l)}$ of shape $|\tilde{M}_{I,J,s}^{(l)}| = \tilde{\sigma}_{I,J,s}^{(l)}$ with the following properties. First, $C_{\tilde{M}_{I,J}^{(l)}} = C_{\tilde{M}_{I,J,s}^{(l)}}$ for all s. This implies

$$\sum_i (\tilde{\sigma}_{I,J,s}^{(l)})_i = d. \tag{3.1}$$

Next, it follows that $\tilde{M}_{I,J,s}^{(l)} \geq \tilde{M}_{I,J}^{(l)}$ for all s. As mentioned above, this implies $\tilde{\sigma}_{I,J,s}^{(l)} \geq \tilde{\sigma}_{I,J}^{(l)}$ and in particular

$$(\tilde{\sigma}_{I,J,s}^{(l)})_1 \geq (\tilde{\sigma}_{I,J}^{(l)})_1 \geq t. \tag{3.2}$$

Note, that 3.1 and 3.2 ensure that the double standard tableaux $\tilde{M}_{I,J,s}^{(l)}$ are in $B_t^{(d)}$. But this proves the statement, because we can write f as a linear combination of these double standard tableaux

$$\begin{aligned}
f &= \sum_{I,J} f_{I,J} m_{I,J} \\
&= \sum_{I,J} \lambda_{I,J} \sum_{l} M_{I,J}^{(l)} m_{I,J} \\
&= \sum_{I,J} \sum_{l} \lambda_{I,J} M_{I,J}^{(l)} m_{I,J} \\
&= \sum_{I,J} \sum_{l} \lambda_{I,J} \tilde{M}_{I,J}^{(l)} \\
&= \sum_{I,J} \sum_{l} \lambda_{I,J} \sum_{s} \mu_s \tilde{M}_{I,J,s}^{l} \\
&= \sum_{I,J} \sum_{l} \sum_{s} \lambda_{I,J} \mu_s \tilde{M}_{I,J,s}^{l}.
\end{aligned}$$

□

CHAPTER 4

Short polynomials in determinantal ideals

In this chapter, we tackle the following problem. What is the smallest number of summands of a nonzero polynomial in a determinantal ideal? We establish a bound from below: I_t does not contain a nonzero polynomial with at most $\frac{t!}{2}$ summands. However, there is hope that this bound can be improved.
The chapter starts with a general discussion of the proof strategy. We specalize to determinantal ideals in the second section. Afterwards, we study the determinantal ideal generated by maximal 2-minors and generalize our approach in the last section.

1. Definitions

In this section, we explain our strategy for understanding short polynomials in classical determinantal ideals. Since our ideas hold in greater generality, we will study the following broader setting. Choose homogeneous polynomials $f_1, \dots, f_r \in K[x_0 \dots, x_n]$ of degree t. We use the notation $f_i = \sum_{\beta \in \mathbb{N}_t^{n+1}} f_{i,\beta} x^\beta$ for all $i = 1, \dots, r$. Fix a natural number $d \in \mathbb{N}_0$. We want to describe the shortest number of summands of a nonzero polynomial in the $(t+d)$-th homogeneous component of the ideal $I = \langle f_1, \dots, f_r \rangle$.

We associate the following linear forms to $I^{(t+d)}$.

Definition 66. *We define the polynomials*

$$p_\alpha := \sum_{i=1}^{r} \sum_{\substack{\beta \in \mathbb{N}_t^{n+1}, \gamma \in \mathbb{N}_d^{n+1} \\ \beta + \gamma = \alpha}} f_{i,\beta} x_{i,\gamma} \in K[(x_{i,\gamma})_{\substack{i=1,\dots,r \\ \gamma \in \mathbb{N}_d^{n+1}}}],$$

for all $\alpha \in \mathbb{N}_{t+d}^{n+1}$.

Lemma 67. *The* p_α *are the coefficients of polynomials in* $I^{(t+d)}$*, to be precise*

$$I^{(t+d)} = \left\{ \sum_{\alpha \in \mathbb{N}_{t+d}^{n+1}} p_\alpha(g) x^\alpha : g \in K^{r |\mathbb{N}_d^{n+1}|} \right\}.$$

Proof. Every $f \in I^{(t+d)}$ is of the form $f = \sum_{i=1}^r g_i f_i$ with $g_i = \sum_{\alpha \in \mathbb{N}_0^{n+1}} g_{i,\alpha} x^\alpha$. Without loss of generality, we can assume that all g_i are homogeneous of degree d.

This can be seen as follows. Writing f in the monomial basis yields

$$\begin{aligned} f &= \sum_{i=1}^{r} g_i f_i \\ &= \sum_{i=1}^{r} \left(\sum_{\alpha \in \mathbb{N}_0^{n+1}} g_{i,\alpha} x^\alpha \right) \left(\sum_{\beta \in \mathbb{N}_t^{n+1}} f_{i,\beta} x^\beta \right) \\ &= \sum_{\alpha \in \mathbb{N}_{t+d}^{n+1}} \left(\sum_{i=1}^{r} \sum_{\substack{\beta \in \mathbb{N}_t^{n+1}, \gamma \in \mathbb{N}_0^{n+1} \\ \beta+\gamma=\alpha}} f_{i,\beta} g_{i,\gamma} \right) x^\alpha. \end{aligned}$$

The coefficient $\sum_{i=1}^{r} \sum_{\substack{\beta \in \mathbb{N}_t^{n+1}, \gamma \in \mathbb{N}_0^{n+1} \\ \beta+\gamma=\alpha}} f_{i,\beta} g_{i,\gamma}$ vanishes if $\alpha \notin \mathbb{N}_{t+d}^{n+1}$, because f is homogeneous of degree $t+d$. And since $g_{i,\gamma}$ for $\gamma \notin \mathbb{N}_d^{n+1}$ only occures in these expressions, we can assume that $g_{i,\gamma} = 0$ for $\gamma \notin \mathbb{N}_d^{n+1}$.
We calculate

$$\begin{aligned} \sum_{i=1}^{r} g_i f_i &= \sum_{i=1}^{r} \left(\sum_{\alpha \in \mathbb{N}_d^{n+1}} g_{i,\alpha} x^\alpha \right) \left(\sum_{\beta \in \mathbb{N}_t^{n+1}} f_{i,\beta} x^\beta \right) \\ &= \sum_{\alpha \in \mathbb{N}_{t+d}^{n+1}} \left(\sum_{i=1}^{r} \sum_{\substack{\beta \in \mathbb{N}_t^{n+1}, \gamma \in \mathbb{N}_d^{n+1} \\ \beta+\gamma=\alpha}} f_{i,\beta} g_{i,\gamma} \right) x^\alpha \\ &= \sum_{\alpha \in \mathbb{N}_{t+d}^{n+1}} p_\alpha(g) x^\alpha, \end{aligned}$$

where $g = (g_{i,\alpha}) \in K^{r|\mathbb{N}_d^{n+1}|}$, which proves "$\subseteq$". We get the other inclusion by performing the previous computation backwards. ☐

Example 68. *Consider $I^{(4)}$, where $I = \langle x^2+xy+y^2 \rangle \subseteq K[x,y]$. We associate the linear forms*

$$\begin{aligned} p_{(4,0)} &= x_{(2,0)} \\ p_{(3,1)} &= x_{(2,0)} + x_{(1,1)} \\ p_{(2,2)} &= x_{(2,0)} + x_{(1,1)} + x_{(0,2)} \\ p_{(1,3)} &= x_{(1,1)} + x_{(0,2)} \\ p_{(0,4)} &= x_{(0,2)} \end{aligned}$$

to it. Then Lemma 67 states that every polynomial in $I^{(4)}$ has the form

$$p_{(4,0)}x^4 + p_{(3,1)}x^3y + p_{(2,2)}x^2y^2 + p_{(1,3)}xy^3 + p_{(0,4)}y^4,$$

which we can write concretly as

$$g_{(2,0)}x^4 + (g_{(2,0)}+g_{(1,1)})x^3y + (g_{(2,0)}+g_{(1,1)}+g_{(0,2)})x^2y^2 + (g_{(1,1)}+g_{(0,2)})xy^3 + g_{(0,2)}y^4,$$

for $g = (g_\alpha) \in K^{\mathbb{N}_2^2}$.

Lemma 67 shows that the p_α are interesting for us: The existence of a polynomial $f \in I^{(t+d)}$ with s summands is equivalent to the existence of a $g \in K^{r|\mathbb{N}_d^{n+1}|}$ such that $p_\alpha(g)$ vanishes for exactly $|\mathbb{N}_{t+d}^{n+1}| - s$ indices $\alpha \in \mathbb{N}_{t+d}^{n+1}$. The next lemma reformulates this observation.

Lemma 69. *$I^{(t+d)}$ does not contain a nonzero polynomial with at most s summands if and only if for all $S \subseteq \mathbb{N}_{t+d}^{n+1}$ with $|S| = |\mathbb{N}_{t+d}^{n+1}| - s$*

$$\operatorname{span}\{p_\alpha : \alpha \in S\} = \operatorname{span}\left\{p_\alpha : \alpha \in \mathbb{N}_{t+d}^{n+1}\right\}.$$

Proof. For all $\alpha \in \mathbb{N}_{t+d}^{n+1}$, we define the vector space V_α to be the kernel of the linear map

$$\begin{aligned} K^{r|\mathbb{N}_d^{n+1}|} &\to K \\ g &\mapsto p_\alpha(g). \end{aligned}$$

Because of the dimension formula we have $\dim V_\alpha = r|\mathbb{N}_d^{n+1}| - 1$ if $p_\alpha \neq 0$ and $\dim V_\alpha = r|\mathbb{N}_d^{n+1}|$ otherwise. Fix the standard inner product on $K^{r|\mathbb{N}_d^{n+1}|}$. By the definition of p_α, its coordinates generate $V_\alpha^\perp$. Thus the following equalities hold.

$I^{(t+d)}$ does not contain a nonzero polynomial with at most s summands

$$\Leftrightarrow \bigcap_{\alpha\in S} V_\alpha = \bigcap_{\alpha\in\mathbb{N}_{t+d}^{n+1}} V_\alpha \text{ for all } S \subseteq \mathbb{N}_{t+d}^{n+1}, |S| = |\mathbb{N}_{t+d}^{n+1}| - s$$

$$\Leftrightarrow \sum_{\alpha\in S} V_\alpha^\perp = \sum_{\alpha\in\mathbb{N}_{t+d}^{n+1}} V_\alpha^\perp \text{ for all } S \subseteq \mathbb{N}_{t+d}^{n+1}, |S| = |\mathbb{N}_{t+d}^{n+1}| - s$$

$$\Leftrightarrow \operatorname{span}\{p_\alpha : \alpha \in S\} = \operatorname{span}\left\{p_\alpha : \alpha \in \mathbb{N}_{t+d}^{n+1}\right\} \text{ for all } S \subseteq \mathbb{N}_{t+d}^{n+1}, |S| = |\mathbb{N}_{t+d}^{n+1}| - s$$

This proves the lemma. ☐

Example 70. *Let us apply Lemma 69 to $I^{(4)}$ from Example 68. We have*

$$\operatorname{span}\{p_{(4,0)}, p_{(3,1)}, p_{(2,2)}, p_{(1,3)}, p_{(0,4)}\} = \operatorname{span}\{x_{(2,0)}, x_{(1,1)}, x_{(0,2)}\}.$$

In particular, the dimension of this vector space is 3. Let $s = 2$. A priori, one might think that $I^{(4)}$ does not contain a polynomial with two summands, but the Lemma says something different. Consider the set $S := \{(4,0),(2,2),(1,3)\}$ with $|S| = 3 = |\mathbb{N}_4^2| - s$. The dimension of $\operatorname{span}\{p_{(4,0)}, p_{(2,2)}, p_{(1,3)}\}$ is 2, in particular

$$\operatorname{span}\{p_\alpha : \alpha \in S\} \neq \operatorname{span}\left\{p_\alpha : \alpha \in \mathbb{N}_{t+d}^{n+1}\right\}.$$

By Lemma 69, this means that $I^{(4)}$ contains a binomial. And indeed, we find

$$x^4 - xy^3 = (x^2 - xy)(x^2 + xy + y^2) \in I^{(4)}.$$

However, $I^{(4)}$ does not contain a monomial, because the vector spaces

$$\begin{gathered} \operatorname{span}\{p_{(4,0)}, p_{(3,1)}, p_{(2,2)}, p_{(1,3)}\} \\ \operatorname{span}\{p_{(4,0)}, p_{(3,1)}, p_{(2,2)}, p_{(0,4)}\} \\ \operatorname{span}\{p_{(4,0)}, p_{(3,1)}, p_{(1,3)}, p_{(0,4)}\} \\ \operatorname{span}\{p_{(4,0)}, p_{(2,2)}, p_{(1,3)}, p_{(0,4)}\} \\ \operatorname{span}\{p_{(3,1)}, p_{(2,2)}, p_{(1,3)}, p_{(0,4)}\} \end{gathered}$$

all have dimension 3.

We give a slightly different version of Lemma 69. It is the key lemma for all further considerations.

LEMMA 71. *$I^{(t+d)}$ does not contain a nonzero polynomial with at most $s-1$ summands if and only if for every $S \subseteq \mathbb{N}^{n+1}_{t+d}$ with $|S| = |\mathbb{N}^{n+1}_{t+d}| - s + 2$ and a distinguished $\beta \in S$, there exists a linear combination $\sum_{\alpha \in S} r_\alpha p_\alpha = 0$ with $r_\beta \neq 0$.*

We close this section with two corollaries.

PROPOSITION 72. *Suppose that the smallest number of summands of a nonzero polynomial in $I^{(t+d)}$ is equal to s. Then the dimension of* $\operatorname{span}\left\{p_\alpha : \alpha \in \mathbb{N}^{n+1}_{t+d}\right\}$ *is at most $|\mathbb{N}^{n+1}_{t+d}| - s + 1$.*

Proof. Choose a subset $S \subseteq \mathbb{N}^{n+1}_{t+d}$ of cardinality $|\mathbb{N}^{n+1}_{t+d}| - s + 1$. Then the linear forms $\{p_\alpha \colon \alpha \in S\}$ generate $\operatorname{span}\left\{p_\alpha : \alpha \in \mathbb{N}^{n+1}_{t+d}\right\}$, by Lemma 69. This implies

$$\dim \operatorname{span}\left\{p_\alpha : \alpha \in \mathbb{N}^{n+1}_{t+d}\right\} \leq |S| = |\mathbb{N}^{n+1}_{t+d}| - s + 1.$$

□

PROPOSITION 73. *Let s be the minimal number of summands of nonzero polynomials in $I^{(t+d)}$, where I is not the zero ideal. This number satisfies*

$$s \geq \min_{\substack{x_{i,\gamma} \in \mathbb{N}^{n+1}_d \\ \exists \alpha : x_{i,\gamma} \in \operatorname{supp}(p_\alpha)}} |\{\alpha : x_{i,\gamma} \in \operatorname{supp}(p_\alpha)\}|.$$

Proof. Since I is not the zero ideal, there exists a variable $x_{i,\gamma}$ such that $\{\alpha : x_{i,\gamma} \in \operatorname{supp}(p_\alpha)\} \neq \emptyset$. We suppose that $|\{\alpha : x_{i,\gamma} \in \operatorname{supp}(p_\alpha)\}| < s$. Fix a polynomial p_β, such that $x_{i,\gamma} \in \operatorname{supp}(p_\beta)$. Moreover, we choose a subset $S \subseteq \mathbb{N}^{n+1}_{t+d}$, such that $|S| = |\mathbb{N}^{n+1}_{t+d}| - s + 2$ and $\{\alpha : x_{i,\gamma} \in \operatorname{supp}(p_\alpha)\} \cap S = \{\beta\}$. This is possible since $|\{\alpha : x_{i,\gamma} \in \operatorname{supp}(p_\alpha)\} \setminus \{\beta\}| \leq s - 2$.
But this implies $r_\beta = 0$ for every relation $\sum_{\alpha \in S} r_\alpha p_\alpha = 0$, because there is no other polynomial p_α with $\alpha \in S$ such that $x_{i,\gamma} \in \operatorname{supp}(p_\alpha)$. This is a contradiction, by Lemma 71 above. □

2. Determinantal ideals

We specialize to determinantal ideals. Let $X = (x_{ij})$ be an $m \times n$-matrix of indeterminants over K and $K[X]$ the polynomial ring over K with indeterminants from X. A *determinantal ideal* is

$$I_t := \langle t \times t\text{-minors of } X \rangle,$$

where $0 < t \leq \min(m, n)$.

We introduce the following notation. For subsets $I \subseteq [m], J \subseteq [n]$ of cardinality t, we define $S_{I,J} := \{\sigma : I \to J \text{ bijective}\}$. Elements of $S_{I,J}$ are called *permutations*. The *signum* of a $\sigma \in S_{I,J}$ is $\operatorname{sgn}(\sigma) := \operatorname{sgn}(\psi \circ \sigma \circ \phi)$, where $\phi : [t] \to I, \psi : J \to [t]$ are bijective and order preserving. The *permutation matrix* $E_\sigma \in \{0,1\}^{m \times n}$ of σ has (i,j)-entry equal to 1 if and only if $i \in I$ and $\sigma(i) = j$.

We want to use methods from the last section to determine the smallest number of summands of a nonzero polynomial in an homogeneous component $I_t^{(t+d)}$, for a

$d \in \mathbb{N}_0$. To do so, we consider the polynomials p_α, in the case of determinantal ideals

$$p_\alpha = \sum_{\substack{\sigma \in S_{I,J} \\ E_\sigma \leq \alpha}} \operatorname{sgn}(\sigma) x_{\alpha - E_\sigma, I, J},$$

where $E_\sigma \leq \alpha$ is defined entrywise. To see this, we copy the proof from Lemma 67. We showed that every polynomial in $I_t^{(t+d)}$ is of the form $\sum_{\substack{I \subseteq [m], J \subseteq [n] \\ |I|=|J|=t}} g_{I,J} \det(X_{I,J})$, where $g_{I,J} = \sum_{\alpha \in \mathbb{N}_d^{m \times n}} g_{\alpha,I,J} x^\alpha$ is homogeneous of degree d. We calculate

$$\begin{aligned}
\sum_{\substack{I \subseteq [m], J \subseteq [n] \\ |I|=|J|=t}} g_{I,J} \det(X_{I,J}) &= \sum_{\substack{I \subseteq [m], J \subseteq [n] \\ |I|=|J|=t}} \left(\sum_{\alpha \in \mathbb{N}_d^{m \times n}} g_{\alpha,I,J} x^\alpha \right) \left(\sum_{\sigma \in S_{I,J}} \operatorname{sgn}(\sigma) \prod_{i \in I} x_{i,\sigma(i)} \right) \\
&= \sum_{\substack{I \subseteq [m], J \subseteq [n] \\ |I|=|J|=t}} \sum_{\alpha \in \mathbb{N}_d^{m \times n}} \sum_{\sigma \in S_{I,J}} \operatorname{sgn}(\sigma) g_{\alpha,I,J} x^{E_\sigma + \alpha} \\
&= \sum_{\alpha \in \mathbb{N}_{t+d}^{m \times n}} \left(\sum_{\substack{\sigma \in S_{I,J} \\ E_\sigma \leq \alpha}} \operatorname{sgn}(\sigma) g_{\alpha - E_\sigma, I, J} \right) x^\alpha.
\end{aligned}$$

In the next sections, we use Lemma 71 to find a bound on the minimal number of summands of a nonzero polynomial in I_t.

3. Short polynomials in I_2

In this section, we study the special case of $I_2 \subseteq K[X]$, where X is a $2 \times n$ generic matrix.

We fix a degree $d \geq 0$ and use the notation $x_{\beta,J} := x_{\beta,\{1,2\},J}$ for the variables and $S_J := S_{\{1,2\},J}$ for the permutations.

Lemma 71 suggests to study the relations between the p_α.

Definition 74. *For all $A \subseteq \mathbb{N}_{2+d}^{2 \times n}$, we define the vector spaces*

$$R_A := \left\{ r = (r_\alpha)_{\alpha \in A} \in K^A \colon \sum_{\alpha \in A} r_\alpha p_\alpha = 0 \right\}.$$

In particular, $R := R_{\mathbb{N}_{2+d}^{2 \times n}}$ is a vector subspace of $K^{\mathbb{N}_{2+d}^{2 \times n}}$.

We study R via the following graph.

Definition 75. *We define an undirected graph $G = (V, E)$ as follows. The vertices are $V := \mathbb{N}_{2+d}^{2 \times n}$. The edges E are given by $\{\alpha, \beta\} \subseteq V$ such that there exists a permutation $\sigma \in S_J$ with $\beta = \alpha - E_\sigma + E_{\sigma \circ \tau_J}$, where $\tau_J \in S_J$ is given by $\tau_J(1) > \tau_J(2)$.*

The following lemma motivates the previous definition.

Lemma 76. *There exists an edge $\{\alpha, \beta\}$ in G if and only if $p_\alpha \neq p_\beta$ share a variable.*

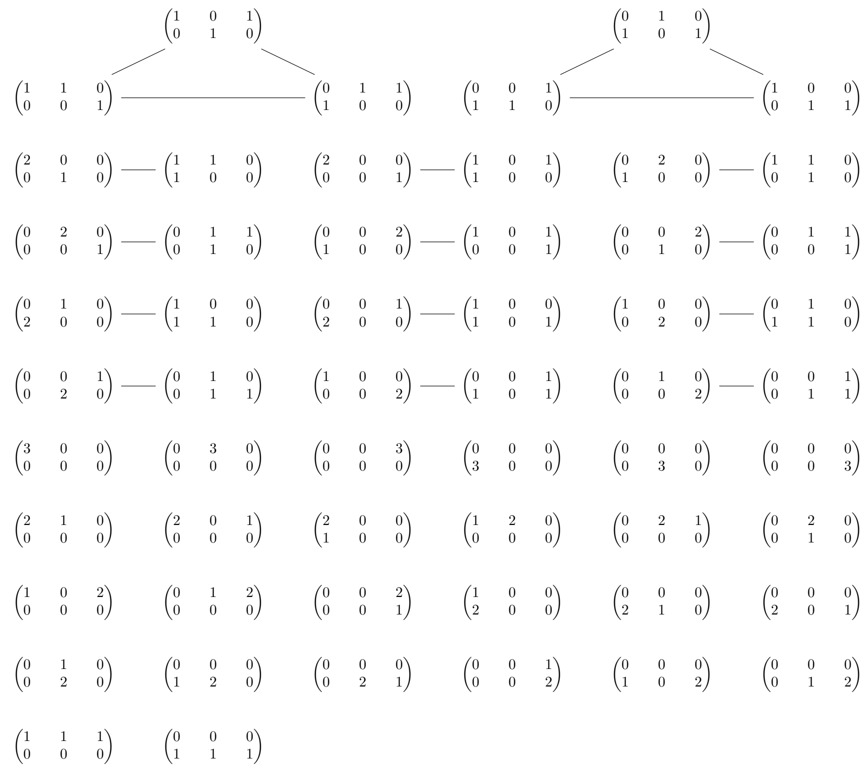

FIGURE 1. The graph G for $I_2^{(3)}$, where X is a 2×3-matrix.

Proof. Let $\{\alpha, \beta\} \in E$. Then there exists a $\sigma \in S_J$ with $\beta = \alpha - E_\sigma + E_{\sigma\circ\tau_J}$. Since for given $i \in [2], j \in J$, either the (i,j)-entry of E_σ or the (i,j)-entry of $E_{\sigma\circ\tau_J}$ is 1, it follows that $E_\sigma \leq \alpha$ and $E_{\sigma\circ\tau_J} \leq \beta$. Therefore we get

$$x_{\alpha - E_\sigma, J} = x_{\beta - E_{\sigma\circ\tau_J}, J} \in \operatorname{supp}(p_\alpha) \cap \operatorname{supp}(p_\beta).$$

Next, suppose that $p_\alpha \neq p_\beta$ share a variable $x_{\gamma,J}$. Since $|S_2| = 2$, $x_{\gamma,J}$ can only be containend in the polynomials $p_{\gamma + E_{id_J}}$ and $p_{\gamma + E_{\tau_J}}$, where $id_J \in S_J$ denotes the identity. Thus without loss of generality, we have $\alpha = \gamma + E_{id_J}$ and $\beta = \gamma + E_{\tau_J}$. In particular, we get

$$\alpha - E_{id_J} + E_{id_J\circ\tau_J} = \beta,$$

which means $\{\alpha, \beta\} \in E$. □

We collect three lemmas on R.

LEMMA 77. *We have*

$$R = \bigoplus_A R_A,$$

where the direct sum varies over all connected components $A \subseteq V$ of G.

Proof. First, we show that $R = \sum_A R_A$, where we view R_A as a subspace of $K^{\mathbb{N}_{2+d}^{2\times n}}$. The inclusion $\supseteq$ is clear. Now let $r \in R$, i.e. $\sum_{\alpha \in \mathbb{N}_{2+d}^{2\times n}} r_\alpha p_\alpha = 0$. We consider $\sum_{\alpha\in A} r_\alpha p_\alpha$, where $A \subseteq V$ is an arbitrary connected component of G. By

Lemma 76, no p_α for $\alpha \in A$ shares a variable with a p_β, where β lies in a different connected component of G. This implies $\sum_{\alpha \in A} r_\alpha p_\alpha = 0$.
It remains to show that the sum is direct. This follows from the fact that the connected components of G form a partition of V. $\square$

LEMMA 78. *Let $A \subseteq V$ be a connected component of G and $r \in R_A \setminus \{0\}$. Then r has support A.*

Proof. Let $r_\alpha \neq 0$ and β a vertex in A. Then there exists a path $(\alpha_1, \ldots, \alpha_s)$ such that $\alpha_1 = \alpha$ and $\alpha_s = \beta$. Since $r_{\alpha_1} \neq 0$, it follows that $r_{\alpha_2} \neq 0$, since there exists a variable $x_{\gamma,J}$, which is only contained in the support of p_{α_1} and p_{α_2}. Repeting this step, we get $r_\beta \neq 0$. $\square$

LEMMA 79. *Let $A \subseteq V$ be a connected component of G. Then R_A is generated by $\sum_{\alpha \in A} e_\alpha$, where $(e_\alpha)_\alpha$ denote the standard unit vectors in $K^{\mathbb{N}_{t+d}^{2\times n}}$.*

Proof. Each variable in $\operatorname{supp}(p_\alpha)$ with $\alpha \in A$ is contained in the support of exactly one other polynomial p_β multiplied by -1, where β lies in the same connected component. This implies $\sum_{\alpha \in A} p_\alpha = 0$ and thus $\sum_{\alpha \in A} e_\alpha \in R_A$.
Suppose that there exist two lineary independent relations $r_1, r_2 \in R_A$. Lemma 78 implies $\operatorname{supp}(r_1) = \operatorname{supp}(r_2) = A$. But then we would find a linear combination with a support, which is properly contained in A. This contradicts the previous lemma. Thus $\dim R_A = 1$, which finishes the proof. $\square$

Combining these lemmas, we get the following characterisation of R.

PROPOSITION 80. *The set of sums $\sum_{\alpha \in A} e_\alpha$ for all connected components $A \subseteq V$ of G is a basis of R. In particular, the dimension of R is equal to the number of connected components of G.*

EXAMPLE 81. *Figure 1 shows G for $I_2^{(3)}$, in the case of a 2×3-matrix X. Proposition 80 now says, that this gives us all relations between the p_α, for $\alpha \in \mathbb{N}_3^{2\times 3}$. For example, we get*

$$p_{\left(\begin{smallmatrix}1&0&1\\0&1&0\end{smallmatrix}\right)} + p_{\left(\begin{smallmatrix}1&1&0\\0&0&1\end{smallmatrix}\right)} + p_{\left(\begin{smallmatrix}0&1&1\\1&0&0\end{smallmatrix}\right)} = 0$$
$$p_{\left(\begin{smallmatrix}0&1&0\\1&0&1\end{smallmatrix}\right)} + p_{\left(\begin{smallmatrix}0&0&1\\1&1&0\end{smallmatrix}\right)} + p_{\left(\begin{smallmatrix}1&0&0\\0&1&1\end{smallmatrix}\right)} = 0.$$

from the triangles in G.

We are now able to proof the main theorem of this section.

THEOREM 82. *I_2 does not contain a polynomial with at most two summands.*

Proof. We prove this statement using Lemma 71. For any $S \subseteq \mathbb{N}_{2+d}^{2\times n}$ with $|S| = |\mathbb{N}_{2+d}^{2\times n}| - 2! + 2$ and a distinguished $\beta \in S$, we have to find a linear combination $\sum_{\alpha \in S} \lambda_\alpha = 0$, such that $\lambda_\beta \neq 0$.
Since $|S| = |\mathbb{N}_{2+d}^{2\times n}| - 2! + 2 = |\mathbb{N}_{2+d}^{2\times n}|$, we only have to check this for $S = \mathbb{N}_{2+d}^{2\times n}$. Given an element $\beta \in S$. If β lies in the connected component $A \subseteq V$ of G, we get $p_\beta + \sum_{\alpha \in A\setminus\{\beta\}} p_\alpha = 0$, by Proposition 80. $\square$

The statement of Theorem 82 itself is not interesting, because we proved it already in Corollary 61. The important part is the proof: In contrast to the proof of Corollary 61, we can generalize the new argument to arbitrary I_t.

4. Short polynomials in I_t

In this section, we study relations among the p_α of the ideal $I_t \subseteq K[X]$, where X is a $m \times n$ generic matrix. This leads to the following result: I_t does not contain a nonzero polynomial with at most $\frac{t!}{2}$ summands.

4.1. First definitions.

Fix a degree $d \geq 0$.

Definition 83. *We define the vector space*

$$R := \left\{ r = (r_\alpha)_{\alpha \in \mathbb{N}^{m\times n}_{t+d}} \in K^{\mathbb{N}^{m\times n}_{t+d}} : \sum_{\alpha \in \mathbb{N}^{m\times n}_{t+d}} r_\alpha p_\alpha = 0 \right\}.$$

For $t = 2$, every variable $x_{\gamma,I,J}$ is contained in the support of exactly two polynomials, with different signs. This information is also stored in the graph G: For every variable $x_{\gamma,I,J}$ of a polynomial p_α, we have an edge to the unique other polynomial p_β which also contains the variable $x_{\gamma,I,J}$. For general minors, we define a set $\mathcal{G}$ of graphs in which every graph satisfies this property. Thus we can use arguments similar to those in the case of rank two minors, although the polynomials p_α fail to have this property.
Also, we introduce weights on these graphs to keep track of the signs before variables.

Definition 84. *We define the set $\mathcal{G}$ of the following weighted undirected graphs $G = (V, E, \delta, w)$. Let $V \subseteq \mathbb{N}^{m\times n}_{t+d}$ and fix for every $\alpha \in V$ and for every $\sigma \in S_{I,J}$ such that $E_\sigma \leq \alpha$ a unique permutation $\pi_{\alpha,\sigma} \in S_{I,J} \setminus \{\sigma\}$, such that $\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}} \in V$ and $\pi_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}, \pi_{\alpha,\sigma}} = \sigma$. The set of edges is the set of formal symbols*

$$E := \{e_{\alpha,\sigma} : \sigma \in S_{I,J}, E_\sigma \leq \alpha\},$$

such that $e_{\alpha,\sigma} = e_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}, \pi_{\alpha,\sigma}}$. Also, we define

$$\begin{aligned} \delta : E &\to \mathcal{P}(V) \\ e_{\alpha,\sigma} &\mapsto \{\alpha, \alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}\}. \end{aligned}$$

We think of $e_{\alpha,\sigma} \in E$ as an edge between the two vertices in $\delta(e_{\alpha,\sigma})$. Furthermore, we define the weight function

$$\begin{aligned} w : E &\to K \\ e_{\alpha,\sigma} &\mapsto -\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi_{\alpha,\sigma}). \end{aligned}$$

and require G to be connected.

Proposition 85. *The weight function of a $G \in \mathcal{G}$ is well-defined.*

Proof. For any $e_{\alpha,\sigma} = e_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}, \pi_{\alpha,\sigma}} \in E$

$$\begin{aligned} w(e_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}, \pi_{\alpha,\sigma}}) &= -\operatorname{sgn}(\pi_{\alpha,\sigma})\operatorname{sgn}(\pi_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}, \pi_{\alpha,\sigma}}) \\ &= -\operatorname{sgn}(\pi_{\alpha,\sigma})\operatorname{sgn}(\sigma) \\ &= w(e_{\alpha,\sigma}). \end{aligned}$$

□

The weight function does not depend on the specific choice of an edge $e_{\alpha,\sigma}$, but only on the choice of two adjacent vertices.

LEMMA 86. *For all permutations $\sigma, \tau \in S_{I,J}$ such that* $\operatorname{sgn}(\sigma|_N) = \operatorname{sgn}(\tau|_N)$, *where* $N := \{i \in I : \sigma(i) \neq \tau(i)\}$, *it follows that* $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)$.

Proof. Choose transpositions $\pi_1, \dots, \pi_s$ such that

$$\sigma\tau^{-1} = \pi_1 \cdots \pi_s.$$

Since $(\sigma\tau^{-1})(j) = j$ for all $i \in J \setminus \tau(N)$, we assume without loss of generality that $\pi_k(j) = j$, for all $i \in J \setminus \tau(N)$ and for all $k = 1, \dots, s$. In particular, the $\pi_k|_{\tau(N)}$ are transpositions. Because $\operatorname{sgn}(\sigma|_N) = \operatorname{sgn}(\tau|_N)$, it follows that

$$\begin{aligned}\operatorname{sgn}(\pi_1|_{\tau(N)} \cdots \pi_s|_{\tau(N)}) &= \operatorname{sgn}(\sigma|_N (\tau|_N)^{-1}) \\ &= 1,\end{aligned}$$

so s is even. This implies

$$\begin{aligned}\operatorname{sgn}(\sigma\tau^{-1}) &= \operatorname{sgn}(\pi_1 \cdots \pi_s) \\ &= 1\end{aligned}$$

and therefore $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)$. □

PROPOSITION 87. *All edges $e_{\alpha,\sigma_1}, e_{\alpha,\sigma_2}$ with $\sigma_1 \in S_{I_1,J_1}, \sigma_2 \in S_{I_2,J_2}$ such that $\delta(e_{\alpha,\sigma_1}) = \delta(e_{\alpha,\sigma_2})$ satisfy*

$$w(e_{\alpha,\sigma_1}) = w(e_{\alpha,\sigma_2}).$$

Proof. The equality $\delta(e_{\alpha,\sigma_1}) = \delta(e_{\alpha,\sigma_2})$ means

$$-E_{\sigma_1} + E_{\pi_{\alpha,\sigma_1}} = -E_{\sigma_2} + E_{\pi_{\alpha,\sigma_2}}. \tag{4.1}$$

Define the sets $N_j := \{i \in I_j : \sigma_j(i) \neq \pi_{\alpha,\sigma_j}(i)\}$ for $j = 1, 2$. Let $i \in N_1$, then the i-th row of $-E_{\sigma_1} + E_{\pi_{\alpha,\sigma_1}}$ has one entry equals 1 and one entry equals -1, and so has $-E_{\sigma_2} + E_{\pi_{\alpha,\sigma_2}}$. It follows that $i \in N_2$ and thus $N_1 = N_2$.
We define $N := N_1 = N_2$. Equation 4.1 implies $\sigma_1|_N = \sigma_2|_N$ and $\pi_{\alpha,\sigma_1}|_N = \pi_{\alpha,\sigma_2}|_N$. This can be seen by looking at the entries of the matrices. In particular, we have $\operatorname{sgn}(\sigma_1|_N) = \operatorname{sgn}(\sigma_2|_N)$ and $\operatorname{sgn}(\pi_{\alpha,\sigma_1}|_N) = \operatorname{sgn}(\pi_{\alpha,\sigma_2}|_N)$. This implies $\operatorname{sgn}(\sigma_1) = \operatorname{sgn}(\sigma_2)$ and $\operatorname{sgn}(\pi_{\alpha,\sigma_1}) = \operatorname{sgn}(\pi_{\alpha,\sigma_2})$, by Lemma 86. We calculate

$$\begin{aligned}w(e_{\alpha,\sigma_1}) &= -\operatorname{sgn}(\sigma_1)\operatorname{sgn}(\pi_{\alpha,\sigma_1}) \\ &= -\operatorname{sgn}(\sigma_2)\operatorname{sgn}(\pi_{\alpha,\sigma_2}) \\ &= w(e_{\alpha,\sigma_2}).\end{aligned}$$

□

We are now able to introduce the following notation. We define for all adjacent vertices α, β which are connected by an edge $e_{\alpha,\sigma}$

$$w(\alpha, \beta) = w(\beta, \alpha) := w(e_{\alpha,\sigma}).$$

In particular, w is even independent of the choice of a specific graph $G \in \mathcal{G}$.

4.2. Finding relations. We show that the following set of graphs plays a role similar to the connected components in Section 3.

DEFINITION 88. *We define $\mathcal{G}^*$ to be the set of graphs $G \in \mathcal{G}$, such that for every cycle $\alpha_1, \dots, \alpha_s = \alpha_1$ in G*

$$\prod_{i=1}^{s-1} w(\alpha_i, \alpha_{i+1}) = 1.$$

We prove the following statement in the remainder of this subsection. Given a graph $G = (V, E, \delta, w) \in \mathcal{G}^*$, we can find a relation $r \in R$ with support equals V. The next definition introduces a candidate for this relation. It mimics the algorithm *breadth-first search*. Further information on this algorithm and related data structures can be found in [**19**, Section 22.2].

DEFINITION 89. *For a graph $G = (V, E, \delta, w) \in \mathcal{G}$ and an arbitrary vertex $\beta \in V$, we define $r^{(G,\beta)} \in K^{\mathbb{N}_{t+d}^{m\times n}}$ to be the vector r after processing the algorithm Relation(G, β).*

Algorithm 1: Relation(G, β)

introduce a vector $r \in K^{\mathbb{N}_{t+d}^{m\times n}}$, such that $r_\beta = 1$ and $r_\alpha = 0$ for every $\alpha \neq \beta$;
introduce a queue Q;
Enqueue(Q, β);
while *Q not empty* **do**
    $\alpha :=$*Dequeue(Q);*
    for *permutations $\sigma \in S_{I,J}$ such that $E_\sigma \leq \alpha$* **do**
        if $r_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}} = 0$ **then**
            $r_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}} := w(\alpha, \alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}) r_\alpha$;
            Enqueue$(Q, \alpha - E_\sigma + E_{\pi_{\alpha,\sigma}})$;
        end
    end
end

Algorithm 1 only differs to breadth-first search in the choice of the vector r. Thus Relation(G, β) terminates and $\mathrm{supp}(r^{(G,\beta)}) = V$, since breadth-first search terminates and visits every vertex in a connected graph.

LEMMA 90. *Let $G = (V, E, \delta, w)$ be a graph in $\mathcal{G}$ and let β be a vertex of G. Then every two adjacent vertices $\alpha, \tilde{\alpha} \in V$ satisfy*

$$r_{\tilde{\alpha}}^{(G,\beta)} = w(\alpha, \tilde{\alpha}) r_\alpha^{(G,\beta)}$$

if and only if $G \in \mathcal{G}^*$.

Proof. We write $r := r^{(G,\beta)}$ to simplify the notation.
First, let $G \in \mathcal{G}^*$ and suppose without loss of generality that the algorithm reaches $\tilde{\alpha}$ by walking through a path $(\alpha_1, \ldots, \alpha_s)$, where $\alpha_1 = \alpha$ and $\alpha_s = \tilde{\alpha}$. Because $G \in \mathcal{G}^*$, it follows that

$$\begin{aligned} r_{\tilde{\alpha}} &= r_{\alpha_s} \\ &= w(\alpha_{s-1}, \alpha_s) r_{\alpha_{s-1}} \\ &= \ldots \\ &= \left(\prod_{i=1}^{s-1} w(\alpha_i, \alpha_{i+1}) \right) r_{\alpha_1} \\ &= w(\alpha_s, \alpha_1) r_{\alpha_1} \\ &= w(\alpha, \tilde{\alpha}) r_\alpha. \end{aligned}$$

Note that $w(\alpha_s, \alpha_1) = w(\alpha_s, \alpha_1)^{-1}$, since w takes values in $\{-1, 1\}$.
To prove the converse, let $(\alpha_1, \dots, \alpha_{s+1})$ be a cycle in G, i.e. $\alpha_1 = \alpha_{s+1}$. We calculate

$$\begin{aligned} w(\alpha_1, \alpha_s) r_{\alpha_1} &= r_{\alpha_s} \\ &= w(\alpha_{s-1}, \alpha_s) r_{\alpha_{s-1}} \\ &= \dots \\ &= \left(\prod_{i=1}^{s-1} w(\alpha_i, \alpha_{i+1}) \right) r_{\alpha_1}. \end{aligned}$$

It follows

$$\prod_{i=1}^{s} w(\alpha_i, \alpha_{i+1}) = 1,$$

because $w(\alpha_1, \alpha_s) = w(\alpha_s, \alpha_1)$ and $r_{\alpha_1} \neq 0$. □

PROPOSITION 91. *Let $G = (V, E, \delta, w)$ be a graph in $\mathcal{G}^*$ and let β be a vertex of G. Then $r^{(G,\beta)}$ is an element of R.*

Proof. We write $r := r^{(G,\beta)}$ to simplify the notation.
Consider a variable $x_{\gamma,I,J}$, such that the set $A := \{\alpha \in V : x_{\gamma,I,J} \in \operatorname{supp}(p_\alpha)\}$ is not empty. We claim that for every vertex $\alpha \in A$ there exists an unique vertex $\tilde{\alpha} \in A$ that is adjacent to α. Let $\sigma \in S_{I,J}$ such that $\gamma = \alpha - E_\sigma$. By the definition of $\mathcal{G}$ there exists a unique permutation $\pi_{\alpha,\sigma}$ such that $\tilde{\alpha} := \alpha - E_\sigma + E_{\pi_{\alpha,\sigma}} \in V$ and $\{\alpha, \tilde{\alpha}\} \in E$. In particular, we have $\tilde{\alpha} \in A$. To prove uniqueness, suppose there are given two edges $\{\alpha, \tilde{\alpha}_1\}$ and $\{\alpha, \tilde{\alpha}_2\}$ with this property. This means that there exist permutations σ_1, σ_2, such that $\alpha_1 = \alpha - E_{\sigma_1} + E_{\pi_{\alpha,\sigma_1}}$, $\alpha_2 = \alpha - E_{\sigma_2} + E_{\pi_{\alpha,\sigma_2}}$ and $\gamma = \alpha - E_{\sigma_1}$ and $\gamma = \alpha - E_{\sigma_2}$. But then it follows that $\sigma_1 = \sigma_2$ and thus $\pi_{\alpha,\sigma_1} = \pi_{\alpha,\sigma_2}$. This implies $\alpha_1 = \alpha_2$.
Because of the previous claim, we can write A as a collection of pairwise distinct polynomials $p_{\alpha_1}, \dots, p_{\alpha_s}, p_{\tilde{\alpha}_1}, \dots, p_{\tilde{\alpha}_s}$, such that only α_i and $\tilde{\alpha}_i$ are adjacent. Also we define permutations $\sigma_i, \tilde{\sigma}_i \in S_{I,J}$ such that $\gamma = \alpha_i - E_{\sigma_i}$ and $\gamma = \tilde{\alpha}_i - E_{\tilde{\sigma}_i}$ for all $i = 1, \dots, s$. Then the coefficient of $x_{\gamma,I,J}$ in the linear combination $\sum_{\alpha \in V} r_\alpha p_\alpha$ equals

$$\begin{aligned} \sum_{i=1}^{s} r_{\alpha_i} \operatorname{sgn}(\sigma_i) + r_{\tilde{\alpha}_i} \operatorname{sgn}(\tilde{\sigma}_i) &= \sum_{i=1}^{s} r_{\alpha_i} \operatorname{sgn}(\sigma_i) + w(\alpha_i, \tilde{\alpha}_i) r_{\alpha_i} \operatorname{sgn}(\tilde{\sigma}_i) \\ &= \sum_{i=1}^{s} r_{\alpha_i} \operatorname{sgn}(\sigma_i) - \operatorname{sgn}(\sigma_i) \operatorname{sgn}(\tilde{\sigma}_i) r_{\alpha_i} \operatorname{sgn}(\tilde{\sigma}_i) \\ &= \sum_{i=1}^{s} r_{\alpha_i} \operatorname{sgn}(\sigma_i) - \operatorname{sgn}(\sigma_i) r_{\alpha_i} \\ &= 0, \end{aligned}$$

because of Lemma 90. It follows $\sum_{\alpha \in V} r_\alpha p_\alpha = 0$. □

Unfortunately, it is not easy to give a full description of R, as in the case of $t = 2$. And because our considerations are sufficient for the next section, we close our studies of the interaction between $\mathcal{G}^*$ and R with the following question.

QUESTION 92. *Generate the relations $r^{(G,\beta)}$ the vector space R?*

4.3. I_t does not contain a nonzero polynomial with at most $\frac{t!}{2}$ summands. We proof the statement in the title of this subsection. The proof uses Lemma 71, and we construct the relations for this criterion with Proposition 91. This reduces the proof of this statement to understanding graphs in $\mathcal{G}^*$.

LEMMA 93. *Let $S \subseteq \mathbb{N}_{t+d}^{m\times n}$ such that $|S| = |\mathbb{N}_{t+d}^{m\times n}| - \frac{t!}{2} + 1$ and choose an element $\beta \in S$. Then there exists a graph $G = (V, E, \delta, w) \in \mathcal{G}^*$ with $\beta \in V \subseteq S$.*

Proof. We consider the algorithm Graph(S, β), which is almost breadth-first search. Therefore this algorithm terminates and every step is a priori well-defined

Algorithm 2: Graph(S, β)

1 V:=$\emptyset$;
2 put β into V;
3 introduce a queue Q;
4 Enqueue(Q, β);
5 **while** *Q not empty* **do**
6   α :=Dequeue(Q);
7   **for** *permutations $\sigma \in S_{I,J}$ such that $E_\sigma \leq \alpha$* **do**
8   **if** *there exists a permutation $\tau \in S_{I,J} \setminus \{\sigma\}$, such that $\alpha - E_\sigma + E_\tau \in V$ and $\pi_{\alpha - E_\sigma + E_\tau, \tau} = \sigma$* **then**
9        $\pi_{\alpha,\sigma} := \tau$;
10   **else**
11   choose a permutation $\pi_{\alpha,\sigma} \in S_{I,J}$ such that $\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}} \in S$ and $\text{sgn}(\sigma) \neq \text{sgn}(\pi_{\alpha,\sigma})$;
12   put $\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}$ into V;
13   Enqueue$(Q, \alpha - E_\sigma + E_{\pi_{\alpha,\sigma}})$;
14   **end**
15   **end**
16 **end**

except for line 11. For any $\alpha \in S$ and $\sigma \in S_{I,J}$ such that $E_\sigma \leq \alpha$, there are $\frac{t!}{2}$ permutations $\pi_{\alpha,\sigma} \in S_{I,J}$ with $\text{sgn}(\pi_{\alpha,\sigma}) \neq \text{sgn}(\sigma)$, because of Lemma 47. Since $|\mathbb{N}_{t+d}^{m\times n} \setminus S| = \frac{t!}{2} - 1$, we can choose such a permutation $\pi_{\alpha,\sigma}$ in line 11, which satisfies $\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}} \in S$.
The algorithm Graph(S, β) defines for every $\alpha \in V$ and $E_\sigma \leq \alpha$ with $\sigma \in S_{I,J}$ a permutation $\pi_{\alpha,\sigma} \in S_{I,J}$ such that $\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}} \in V$. Line 9 garanties that $\pi_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}} = \sigma$. We define a set of formal symbols

$$E := \{e_{\alpha,\sigma} : \sigma \in S_{I,J}, E_\sigma \leq \alpha\},$$

such that $e_{\alpha,\sigma} = e_{\alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}, \pi_{\alpha,\sigma}}$. Also, we define the maps

$$\begin{aligned} \delta : E &\to \mathcal{P}(V) \\ e_{\alpha,\sigma} &\mapsto \{\alpha, \alpha - E_\sigma + E_{\pi_{\alpha,\sigma}}\}. \end{aligned}$$

and

$$\begin{aligned} w : E &\to K \\ e_{\alpha,\sigma} &\mapsto -\text{sgn}(\sigma)\,\text{sgn}(\pi_{\alpha,\sigma}). \end{aligned}$$

To show that $G := (V, E, \delta, w)$ is in $\mathcal{G}$, it remains to prove that G is connected. It suffices to show that there exists a path from every $\alpha \in V$ to $\beta \in V$. First, suppose that $\alpha \neq \beta$. Choose $\alpha_1 \in V$, such that the algorithm sets α in V while α_1 is processed (lines 11-13). This means, that α_1 and α are adjacent in G, since $\alpha = \alpha_1 + E_\sigma - E_{\pi_{\alpha_1,\sigma}}$ for an $E_\sigma \leq \alpha_1$. Repeating this step, we get a path $\alpha, \alpha_1, \alpha_2, \ldots, \beta$. It remains to prove that $G \in \mathcal{G}^*$. Let $e_{\alpha,\sigma}$ be an edge of G with $\delta(e_{\alpha,\sigma}) = \{\alpha, \tilde{\alpha}\}$ and suppose the algorithm sets $\tilde{\alpha}$ into the queue while processing α. With a view towards line 11, we calculate

$$w(e_{\alpha,\sigma}) = -\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi_{\alpha,\sigma}) = 1.$$

In particular, the weight of every edge in G is 1. Thus for any cycle $\alpha_1, \ldots, \alpha_s = \alpha_1$ in G

$$\prod_{i=1}^{s-1} w(\alpha_i, \alpha_{i+1}) = \prod_{i=1}^{s-1} 1 = 1.$$

□

THEOREM 94. *I_t does not contain a nonzero polynomial with at most $\frac{t!}{2}$ summands.*

Proof. Let $S \subseteq \mathbb{N}^{m\times n}_{t+d}$ such that $|S| = |\mathbb{N}^{m\times n}_{t+d}| - \frac{t!}{2} + 1$ and $\beta \in S$. Then there exists a graph $G = (V, E, \delta, w) \in \mathcal{G}^*$ such that $\beta \in V$ and $V \subseteq S$, by Lemma 93. Proposition 91 implies that there exists a relation $r \in R$ such that $\operatorname{supp}(r) = V$. In particular, we have $\operatorname{supp}(r) \subseteq S$ and $r_\beta \neq 0$. But then Lemma 71 states that $I^{(t+d)}$ does not contain a polynomial with at most $\frac{t!}{2}$ summands. Since the choice of $d \geq 0$ was arbitrary, the proposition follows. □

It should be possible to improve this theorem. The reason for this hope relies on the proof of Lemma 93: Every edge in the constructed graph has weight 1. Therefore, the author thinks that it should be possible to improve this bound by systematically including graphs with edges with negative weight.

CONJECTURE 95. *Let $S \subseteq \mathbb{N}^{m\times n}_{t+d}$ such that $|S| = |\mathbb{N}^{m\times n}_{t+d}| - t! + 2$ and choose an element $\beta \in S$. Then there exists a graph $G = (V, E, \delta, w) \in \mathcal{G}^*$ with $\beta \in V \subseteq S$.*

If this conjecture is true, copying the proof of Theorem 94 would yield

CONJECTURE 96. *I_t does not contain a nonzero polynomial with at most $t! - 1$ summands.*

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe.

Ort, Datum, Unterschrift