# Towards a $p$-adic analytic Riemann-Hilbert Correspondence

Finn Wiersig

University of Oxford, St Peter's College

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# Overview

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2. Towards a $p$-adic Riemann-Hilbert correspondence
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# Warm up

## Derived Morita equivalence

- $R$ a ring
- $S$ a ring
- $T$ an $R$-$S$-bimodule

Define Solution and Reconstruction functors

$$\mathrm{Sol}\colon \mathbf{D}(R) \to \mathbf{D}(S)^{\mathrm{op}}, M^\bullet \mapsto \mathrm{R}\operatorname{Hom}_R(M^\bullet, T)$$
$$\mathrm{Rec}\colon \mathbf{D}(S)^{\mathrm{op}} \to \mathbf{D}(R), N^\bullet \mapsto \mathrm{R}\operatorname{Hom}_S(N^\bullet, T).$$

$T$ is tilting if $\rho\colon R \xrightarrow{\cong} R\operatorname{Hom}_S(T, T)$ as complexes of $R$-modules.

### Lemma

If $T$ is tilting, $M^\bullet \xrightarrow{\cong} \mathrm{Rec}(\mathrm{Sol}(M^\bullet))$ for any bounded perfect complex $M^\bullet$ of $R$-modules. $\Rightarrow$ $\mathrm{Sol}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{perf}}(R) \hookrightarrow \mathbf{D}^{\mathrm{b}}(S)^{\mathrm{op}}$.

# Warm up

## Derived Morita equivalence

**Lemma**

If $T$ is tilting, $M^\bullet \xrightarrow{\cong} \operatorname{Rec}(\operatorname{Sol}(M^\bullet))$ for any bounded perfect complex $M^\bullet$ of $R$-modules. $\Rightarrow \operatorname{Sol}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{perf}}(R) \hookrightarrow \mathbf{D}^{\mathrm{b}}(S)^{\mathrm{op}}$.

**Proof.**

$M^\bullet$ perfect $\rightsquigarrow$ suffices to prove the Theorem for $M^\bullet = R$. Then

$$R \to \operatorname{Rec}(\operatorname{Sol}(R)) \cong R\operatorname{Hom}_S(T, T)$$

is an isomorphism because $T$ is tilting. □

The proof of the Lemma applies in great generality.

$X$ complex analytic manifold,
$\mathcal{O}$ sheaf of holomorphic functions on $X$,
$\mathcal{D}^\infty$ sheaf of infinite order differential operators on $X$.

**Example**

For $\mathbb{D} := \mathbb{D}_{\mathbb{C}}(1) \subseteq \mathbb{C}$ the open unit disc with coordinate $z$,

$$\mathcal{D}^\infty(\mathbb{D}) = \left\{ \sum_{\alpha \geq 0} f_\alpha \partial^\alpha : \sum_{\alpha \geq 0} f_\alpha \zeta^\alpha \alpha! \in \mathcal{O}(\mathbb{D} \times \mathbb{C}) \cong \mathcal{O}(T^* \mathbb{D}) \right\}$$

and $\partial \cdot f = \frac{d}{dz} f$ for all $f \in \mathcal{O}(\mathbb{D})$.

# The complex Riemann-Hilbert Correspondence

à la Prosmans-Schneiders

$X$ complex analytic manifold

- $\mathcal{D}^\infty$ sheaf of infinite order differential operators
- $\mathbb{C}_X$ constant sheaf
- $\mathcal{O}$ sheaf of holomorphic functions is a sheaf of $\mathcal{D}^\infty$-$\mathbb{C}_X$-bimodules

Define Solution and Reconstruction functors

$$\mathrm{Sol}\colon\ \mathbf{D}(\mathcal{D}^\infty) \to \mathbf{D}(\mathbb{C}_X)^{\mathrm{op}}, \mathcal{M}^\bullet \mapsto \mathrm{R}\,\underline{\mathcal{H}om}_{\mathcal{D}^\infty}(\mathcal{M}^\bullet, \mathcal{O})$$
$$\mathrm{Rec}\colon\ \mathbf{D}(\mathbb{C}_X)^{\mathrm{op}} \to \mathbf{D}(\mathcal{D}^\infty), \mathcal{F}^\bullet \mapsto \mathrm{R}\,\underline{\mathcal{H}om}_{\mathbb{C}}(\mathcal{F}^\bullet, \mathcal{O}).$$

# The complex Riemann-Hilbert Correspondence

à la Prosmans-Schneiders

**Theorem (Ishimura, 1978; Prosmans-Schneiders, 2000)**

$\mathcal{O}$ is tilting; that is, $\rho\colon \mathcal{D}^\infty \xrightarrow{\cong} \mathrm{R}\,\underline{\mathcal{H}om}_{\mathbb{C}}\,(\mathcal{O}, \mathcal{O})$.

Consider only the $\mathcal{O} \to \mathcal{O}$ such that for all open $U \subseteq X$, the induced $\mathcal{O}(U) \to \mathcal{O}(U)$ are continuous.

***Topological Reconstruction Theorem* (Prosmans-Schneiders, 2000)**

$\mathcal{M}^\bullet \xrightarrow{\cong} \mathrm{Rec}\,(\mathrm{Sol}\,(\mathcal{M}^\bullet))$ for any bounded perfect complex $\mathcal{M}^\bullet$ of $\mathcal{D}^\infty$-modules. $\Rightarrow$ $\mathrm{Sol}\colon\; \mathbf{D}^{\mathrm{b}}_{\mathrm{perf}}\,(\mathcal{D}^\infty) \hookrightarrow \mathbf{D}^{\mathrm{b}}\,(\mathbb{C}_X)^{\mathrm{op}}$.

# The complex Riemann-Hilbert Correspondence

## à la Prosmans-Schneiders

**Topological Reconstruction Theorem** *(Prosmans-Schneiders, 2000)*

$\mathcal{M}^\bullet \xrightarrow{\cong} \mathrm{Rec}\,(\mathrm{Sol}\,(\mathcal{M}^\bullet))$ for any bounded perfect complex $\mathcal{M}^\bullet$ of $\mathcal{D}^\infty$-modules. $\Rightarrow$ $\mathrm{Sol}\colon\ \mathbf{D}^{\mathrm{b}}_{\mathrm{perf}}\,(\mathcal{D}^\infty) \hookrightarrow \mathbf{D}^{\mathrm{b}}\,(\mathbb{C}_X)^{\mathrm{op}}$.

**Remarks**

(i) Given $P \in \mathcal{D}^\infty$, $\mathrm{Sol}\,(\mathcal{D}^\infty/\mathcal{D}^\infty P) \approx$ solutions of $P = 0$. (Related to Hilbert 21st problem.)

(ii) Prosmans-Schneiders' Theorem generalises work of Kashiwara and Mebkhout (1984), and bypasses the usage of the six-functor formalism for holonomic $\mathcal{D}^\infty$.

$p$ prime,
$k/\mathbb{Q}_p$ finite,
$X$ smooth rigid-analytic $k$-variety,
$\widehat{\mathcal{D}}$ sheaf of infinite order differential operators on $X$ (Ardakov-Wadsley).

**Motivation**

Locally analytic representation theory of $p$-adic Lie groups (Schneider-Teitelbaum, 2001); Ardakov (2021): study these via $\widehat{\mathcal{D}}$

**Goal**

Replicate Prosmans-Schneiders' Theorem for $\widehat{\mathcal{D}}$-modules.

### Example

For $\mathbb{D} := \mathbb{D}_k\left(p^0\right)$ the closed unit disc,

$$\widehat{\mathcal{D}}(\mathbb{D}) = \left\{ \sum_{\alpha \geq 0} a_\alpha \partial^\alpha \colon \sum_{\alpha \geq 0} a_\alpha \zeta^\alpha \alpha! \in \mathcal{O}\left(T^* \mathbb{D}^1\right) \cong \mathcal{O}\left(\mathbb{D} \times \mathbb{A}_k^{1,\mathrm{an}}\right) \right\}$$

$$= \varprojlim_{r \in \mathbb{N}} \left\{ \sum_{\alpha \geq 0} a_\alpha \partial^\alpha \colon \sum_{\alpha \geq 0} a_\alpha \zeta^\alpha \alpha! \in \mathcal{O}\left(\mathbb{D} \times \mathbb{D}_k\left(p^r\right)\right) \right\}$$

$$= \left\{ \sum_{\alpha \geq 0} a_\alpha \partial^\alpha \colon \|a_\alpha\| p^{r\alpha} \to 0 \text{ for all } r \in \mathbb{N} \right\}.$$

$X$ smooth rigid-analytic variety over $k$

- $\widehat{\mathcal{D}}$ sheaf of infinite order differential operators
- $k_X$ constant sheaf
- $\mathcal{O}$ sheaf of holomorphic functions is a sheaf of $\widehat{\mathcal{D}}$-$k_X$-bimodules

**Theorem (Ardakov-Ben-Bassat, 2018)**

$\mathcal{O}$ is not tilting since $\rho\colon \widehat{\mathcal{D}} \hookrightarrow \underline{\mathcal{H}om}_k(\mathcal{O}, \mathcal{O})$ is not epi.

**Observation (Berkovich, 2006)**

The augmented de Rham complex

$$0 \to k_X \to \mathcal{O} \xrightarrow{\nabla} \Omega^1 \to \dots$$

is not exact in any degree.

$k$ is too small to contain integrals of differential forms.

**Idea (Fontaine, 1982)**

Replace $k$ by a much larger *field* $B_{\mathrm{dR}}$ *of p-adic periods*.

$B_{\mathrm{dR}}^+ \subseteq B_{\mathrm{dR}}$ discrete valuation ring with residue field $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$.

Scholze introduced relative versions

| | | |
|---|---|---|
| $X \rightsquigarrow X_{\text{proét}}$ | | pro-étale site |
| $k_X \rightsquigarrow \mathbb{B}_{\text{dR}}$ | | de Rham period sheaf (on $X_{\text{proét}}$) |
| $\mathcal{O} \rightsquigarrow \mathcal{O}\mathbb{B}_{\text{dR}}$ | | de Rham period structure sheaf (on $X_{\text{proét}}$). |

Let $\nu \colon X_{\text{proét}} \to X$ denote the canonical projection.

**Theorem (Scholze, 2013)**

*The augmented de Rham complex*

$$0 \to \mathbb{B}_{\text{dR}} \to \mathcal{O}\mathbb{B}_{\text{dR}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\text{dR}} \otimes_{\nu^{-1}\mathcal{O}} \nu^{-1}\Omega^1 \to \dots$$

*on $X_{\text{proét}}$ is exact.*

$X$ smooth rigid-analytic variety over $k$;
$X_{\text{proét}}$ pro-étale site and $\nu\colon X_{\text{proét}} \to X$ canonical projection

- $\widehat{\mathcal{D}}$ sheaf of infinite order differential operators
- $\mathbb{B}_{\mathrm{dR}}$ de Rham period sheaf (on $X_{\text{proét}}$)
- $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ de Rham period structure sheaf (on $X_{\text{proét}}$) is not a sheaf of $\nu^{-1}\widehat{\mathcal{D}}$-modules!

$\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is too big because $\mathbb{B}_{\mathrm{dR}}$ is too big.

Substitute

| | |
|---|---|
| $X \rightsquigarrow X_{\text{proét}}$ | pro-étale site |
| $k_X \rightsquigarrow \mathbb{B}_{\text{la}}$ | locally analytic period sheaf (on $X_{\text{proét}}$) |
| $\mathcal{O} \rightsquigarrow \mathcal{O}\mathbb{B}_{\text{la}}$ | locally analytic period structure sheaf (on $X_{\text{proét}}$). |

Let $\nu\colon X_{\text{proét}} \to X$ denote the canonical projection.

**Theorem (W., 2023)**

*The augmented de Rham complex*

$$0 \to \mathbb{B}_{\text{la}} \to \mathcal{O}\mathbb{B}_{\text{la}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\text{la}} \otimes_{\nu^{-1}\mathcal{O}} \nu^{-1}\Omega^1 \to \dots$$

*on $X_{\text{proét}}$ is exact.*

$X$ smooth rigid-analytic variety over $k$;
$X_{\text{proét}}$ pro-étale site and $\nu \colon X_{\text{proét}} \to X$ canonical projection

- $\widehat{\mathcal{D}}$ sheaf of infinite order differential operators
- $\mathbb{B}_{\mathrm{la}}$ locally analytic period sheaf (on $X_{\text{proét}}$)
- $\mathcal{O}\mathbb{B}_{\mathrm{la}}$ locally analytic period structure sheaf (on $X_{\text{proét}}$) is a sheaf of $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{la}}$-bimodules.

Define Solution and Reconstruction functors

$$\mathrm{Sol}\colon\ \mathbf{D}\left(\widehat{\mathcal{D}}\right) \to \mathbf{D}\left(\mathbb{B}_{\mathrm{la}}\right)^{\mathrm{op}}, \mathcal{M}^{\bullet} \mapsto \mathrm{R}\,\mathrm{Hom}_{\nu^{-1}\widehat{\mathcal{D}}}\left(\nu^{-1}\mathcal{M}^{\bullet}, \mathcal{O}\mathbb{B}_{\mathrm{la}}\right),$$

$$\mathrm{Rec}\colon\ \mathbf{D}\left(\mathbb{B}_{\mathrm{la}}\right)^{\mathrm{op}} \to \mathbf{D}\left(\widehat{\mathcal{D}}\right), \mathcal{F}^{\bullet} \mapsto \mathrm{R}\,\nu_{*}\,\mathrm{R}\,\mathrm{Hom}_{\mathbb{B}_{\mathrm{la}}}\left(\mathcal{F}^{\bullet}, \mathcal{O}\mathbb{B}_{\mathrm{la}}\right).$$

**Conjecture**

*For any $\mathcal{C}$-complex $\mathcal{M}^\bullet \in \mathbf{D}(\widehat{\mathcal{D}})$ (Bode, 2023),*

$$\mathcal{M}^\bullet \xrightarrow{\cong} \mathrm{Rec}\left(\mathrm{Sol}\left(M^\bullet\right)\right).$$

*In particular,* $\mathrm{Sol}\colon \mathbf{D}_{\mathcal{C}}\left(\widehat{\mathcal{D}}\right) \hookrightarrow \mathbf{D}\left(\mathbb{B}_{\mathrm{la}}\right)^{\mathrm{op}}$ *is fully faithful.*

**Theorem (W., 2023)**

$\rho\colon \widehat{\mathcal{D}} \xrightarrow{\cong} \nu_* \underline{\mathcal{H}om}_{\mathbb{B}_{\mathrm{la}}}\left(\mathcal{O}\mathbb{B}_{\mathrm{la}}, \mathcal{O}\mathbb{B}_{\mathrm{la}}\right).$