Reconstructing $\widehat{\mathcal{D}}$ from p-adic Hodge theory

Finn Wiersig
St Peter's College
University of Oxford

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Für Mama und Papa

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Abstract

We construct a solution functor within the context of a still hypothetical p-adic analytic Riemann-Hilbert correspondence. Our approach relies on the overconvergent de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}^{\dagger}$, obtained from an ind-Banach completion of $\mathbb{B}_{\mathrm{inf}}$ along the kernel of Fontaine's map. A key result in this thesis is establishing a $\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger}$-bimodule structure on the period structure sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger}$. Here, $\widehat{\mathcal{D}}$ denotes the sheaf of infinite order differential operators introduced by Ardakov-Wadsley; notably, the analogous statement does not hold for Scholze's $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. We explain how this leads to a solution functor for $\widehat{\mathcal{D}}$-modules and propose conjectures about its compatibility with Scholze's horizontal sections functor and the reconstruction of $\widehat{\mathcal{D}}$-modules from their solutions.

Statement of originality

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree, diploma, certificate or other qualification at the University of Oxford or elsewhere. To the best of my knowledge and belief, the work contained in this thesis is original and my own, unless indicated otherwise. Most of the content in the second and third chapter of this thesis has appeared on arXiv [61].

Finn Wiersig

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Chapter 1

Introduction

1.1 Background

1.1.1 History and Motivation

In his 1900 address at the International Congress of Mathematicians, David Hilbert presented 23 problems that would significantly impact 20th-century mathematics. Among these, he posed a question about the existence of complex differential equations with specific singular points and monodromy groups. Deligne, Kashiwara, and Mebkhout later answered this question with the *Riemann-Hilbert correspondence*, linking *D-modules* and *perverse sheaves*. This connection opened totally new perspectives in representation theory.

Fix a prime p. k denotes a non-Archimedean field of mixed characteristic $(0, p)$. In a series of papers including [48, 49, 50, 51], Schneider and Teitelbaum developed the theory of locally analytic representations of a k-analytic group G in locally convex topological vector spaces over k. These kinds of representations arise naturally in several places in number theory, for example in the p-adic local Langlands program [25, 27, 29]. The author aims to study them with the ideas underpinning the classical Riemann-Hilbert correspondence.

Schneider-Teitelbaum's work inspired Ardakov-Wadsley [7, 6] and Ardakov-Bode-Wadsley [5] to introduce the sheaf $\widehat{\mathcal{D}}$ of infinite order differential operators on a given smooth rigid-analytic variety X over a non-Archimedean field k in mixed characteristic. There already have been strong applications to p-adic representation theory, for example the Beilinson-Bernstein style localisation theorems [7, Theorem E] and [2, Theorem C] as well as the construction of new classes of locally analytic representations [3, Theorem C]. Therefore, we we aim to develop a Riemann-Hilbert correspondence for $\widehat{\mathcal{D}}$-modules.

Notation 1.1.1. κ denotes a finite field of characteristic p, $W(\kappa)$ its ring of (always p-typical) Witt vectors, and k is a finite extension of $k_0 := W(\kappa)[1/p]$. Equip k with the discrete valuation extending the one on k_0.

Notation 1.1.2. X is a fixed smooth rigid-analytic k-variety.

Remark 1.1.3. We omit functional analytic details in this introduction for the sake of clarity. They are expanded upon in the main body of this thesis. References are included along the way.

1.1.2 Prosmans-Schneiders' approach to the Riemann-Hilbert correspondence

The standard proof of the classical Riemann-Hilbert correspondence fails in the p-adic setting, due to the current absence of a six-functor formalism for holonomic $\widehat{\mathcal{D}}$-modules, cf. [1, 17, 5, 18, 19, 32, 33]. Therefore we follow Prosmans-Schneiders' [45] approach in the Archimedean setting. View the structure sheaf $\mathcal{O}$ as a bimodule object for the following two ring-objects: $\widehat{\mathcal{D}}$ and the constant sheaf k_X. This gives

$$\begin{gathered}\mathrm{Sol}\colon\ \mathbf{D}\left(\widehat{\mathcal{D}}\right)^{\mathrm{op}} \to \mathbf{D}\left(k_X\right),\ \ \mathcal{M}^\bullet \mapsto \mathrm{R}\,\underline{\mathcal{H}om}_{\widehat{\mathcal{D}}}\left(\mathcal{M}^\bullet, \mathcal{O}\right),\ \text{and} \\ \mathrm{Rec}\colon\ \mathbf{D}\left(k_X\right) \to \mathbf{D}\left(\widehat{\mathcal{D}}\right)^{\mathrm{op}},\ \mathcal{F}^\bullet \mapsto \mathrm{R}\,\underline{\mathcal{H}om}_{k_X}\left(\mathcal{F}^\bullet, \mathcal{O}\right).\end{gathered}$$

There is a canonical functorial morphism $\mathcal{M}^\bullet \to \mathrm{Rec}\left(\mathrm{Sol}\left(\mathcal{M}^\bullet\right)\right)$ for every complex $\mathcal{M}^\bullet$ of $\widehat{\mathcal{D}}$-modules. We would like it to be an isomorphism for a large collection of $\mathcal{M}^\bullet$, as this would imply that the restriction of Sol to a large full subcategory of $\mathbf{D}\left(\widehat{\mathcal{D}}\right)^{\mathrm{op}}$ is fully faithful. Whilst Prosmans-Schneiders observed that this is indeed the case in the Archimedean setting, Ardakov-Ben-Bassat [4] showed that it is not in the non-Archimedean setting, at least not over our preferred ground field k. This suggests that we consider another context for non-Archimedean geometry.

1.1.3 p-adic Hodge theory

We work within the setting of [54]: Replace the constant sheaf k_X and the structure sheaf $\mathcal{O}$ on X with the positive de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}^+$ and the positive de Rham period structure sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ on the pro-étale site $X_{\text{proét}}$. Then use a local description of the sections of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$, cf. [55], to show that the augmented de Rham complex

$$0 \to \mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \overset{\nabla_{\mathrm{dR}}^+}{\to} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu^{-1}\mathcal{O}} \nu^{-1}\Omega^1 \to \cdots \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu^{-1}\mathcal{O}} \nu^{-1}\Omega^d \to 0$$

is exact. Here $\nu\colon X_{\text{proét}} \to X$ denotes the canonical projection of sites. This is the main input for the construction [54, Theorem 7.6] of Scholze's fully faithful functor

$$\left\{\begin{array}{c}\text{filtered } \mathcal{O}\text{-modules with}\\ \text{integrable connection satisfying}\\ \text{Griffiths transversality}\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\mathbb{B}_{\mathrm{dR}}^{+}\text{-}\textit{local systems}\text{, that is}\\ \text{finite locally free sheaves}\\ \text{of } \mathbb{B}_{\mathrm{dR}}^{+}\text{-modules,}\end{array}\right\}, \quad (1.1.1)$$

which is the first instance of a p-adic de Rham functor.

In the complex analytic setting, the solution functor is closely related to the de Rham functor. Therefore, we expect that Prosmans-Schneiders' ideas can be realised within this geometric context.

1.2 Main results

We need an appropriate variant of the $\widehat{\mathcal{D}}$-k_X-bimodule object $\mathcal{O}$ to apply Prosmans-Schneiders' ideas within the framework of p-adic Hodge theory. Following the work of Scholze, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$ seems to be the natural choice. But $\widehat{\mathcal{D}}$ is a sheaf of *infinite* order differential operators and the $\mathbb{B}_{\mathrm{dR}}^{+}$-linear differential operators on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$ are all of *finite* order. Thus there is no sensible $\nu^{-1}\widehat{\mathcal{D}}$-module structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$. Therefore, we introduce variants of $\mathbb{B}_{\mathrm{dR}}^{+}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$: the *positive overconvergent de Rham period sheaf* [1] $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ and the *positive overconvergent de Rham period structure sheaf* $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$. The following local description of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ in the style of [54, Proposition 6.10] is the main technical input for our constructions.

Theorem 1.2.1 (Theorem 3.5.5)**.** *Assume that X is affinoid and equipped with an étale morphism $X \to \mathbb{T}^d := \operatorname{Sp} k\left\langle T_1^{\pm}, \ldots, T_d^{\pm}\right\rangle$. Denote the induced pro-étale covering introduced in the [54, proof of Corollary 4.7] by $\widetilde{X} \to X$. Then*

$$\varinjlim_{m\geq 0} \mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}} \left\langle \frac{Z_1,\ldots,Z_d}{p^m}\right\rangle \xrightarrow{\cong} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}}, \quad Z_i \mapsto T_i\widehat{\otimes}1 - 1\widehat{\otimes}\left[T_i^{\flat}\right]$$

is an isomorphism of sheaves of $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}}$-algebras.

$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ is small enough to carry an action of $\nu^{-1}\widehat{\mathcal{D}}$, in the following sense.

Theorem 1.2.2 (Theorem 4.2.1)**.** *There exists a $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-bimodule structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ such that the canonical morphism*

$$\nu^{-1}\mathcal{O}\widehat{\otimes}_{k_0}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$$

is a morphism of $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-bimodule objects. It is unique.

[1] A similar construction appeared in [41, Definition 5.1.1]. It would be interesting to establish a precise comparison and study applications of methods developed *loc. cit.* to the theory of $\widehat{\mathcal{D}}$-modules.

$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ is large enough to behave like $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$, in the following sense.

Theorem 1.2.3 (Theorem 4.3.8). *The $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-bimodule structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ induces a $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-linear connection*

$$\nabla_{\mathrm{dR}}^{\dagger,+}\colon \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^{1}.$$

It is integrable, and the associated augmented de Rham complex is strictly exact:

$$0 \to \mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \xrightarrow{\nabla_{\mathrm{dR}}^{\dagger,+}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^{1} \to \cdots \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^{\dim X} \to 0.$$

Remark 1.2.4. Assume that X is affinoid and equipped with an étale morphism $X \to \mathbb{T}^{d} = \operatorname{Sp} k\left\langle T_1^{\pm},\dots,T_d^{\pm}\right\rangle$. The vector fields d/dT_i lift canonically along the étale map $\mathcal{O}\left(\mathbb{T}^{d}\right) \to \mathcal{O}\left(X\right)$ to elements $\partial_i \in \widehat{\mathcal{D}}(X)$. By Theorem 1.2.1, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ has the sections $Z_1,\dots,Z_d$, locally on the proétale site.

$$\partial_i \cdot Z_j := \frac{d}{dZ_i}\left(Z_j\right) = \delta_{ij}$$

then determines the action of $\nu^{-1}\widehat{\mathcal{D}}$ on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$, where δ_{ij} is the Kronecker delta.

1.3 Conjectures

We define the *positive solution functor* via the $\nu^{-1}\widehat{\mathcal{D}}$-bimodule structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$:

$$\mathrm{Sol}^{+}\colon \mathbf{D}\left(\widehat{\mathcal{D}}\right)^{\mathrm{op}} \to \mathbf{D}\left(\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right), \mathcal{M}^{\bullet} \mapsto \mathrm{R}\,\underline{\mathcal{H}om}_{\nu^{-1}\widehat{\mathcal{D}}}\left(\nu^{-1}\mathcal{M}^{\bullet}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right).$$

Following the classical [35, Proposition 4.2.1], the *positive de Rham functor* is

$$\mathrm{dR}^{+}\colon \mathbf{D}\left(\widehat{\mathcal{D}}\right) \to \mathbf{D}\left(\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right), \mathcal{M}^{\bullet} \mapsto \mathrm{Sol}^{+}\left(\mathbb{D}\left(\mathcal{M}^{\bullet}\right)\right)[\dim X].$$

Here, $\mathbb{D}$ is the $\widehat{\mathcal{D}}$-module duality functor, cf. [18, section 5.2]. We expect these constructions to be compatible with Scholze's functor (1.1.1), in the following sense.

Conjecture 1.3.1 (Conjecture 5.3.3). *Consider an $\mathcal{O}$-module with integrable connection $\mathcal{E}$. Equipped with the trivial filtration, Scholze's functor (1.1.1) associates to it a $\mathbb{B}_{\mathrm{dR}}^{+}$-local system $\mathcal{L}$. On the other hand, we view $\mathcal{E}$ as a $\widehat{\mathcal{D}}$-module. Then $\mathrm{dR}^{+}(\mathcal{E})$ is concentrated in degree $-\dim X$ and, functorially in $\mathcal{E}$,*

$$\mathrm{H}^{-\dim X}\left(\mathrm{dR}^{+}\left(\mathcal{E}\right)\right)\otimes_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\mathbb{B}_{\mathrm{dR}}^{+} \xrightarrow{\cong} \mathcal{L}.$$

We would like to reconstruct $\widehat{\mathcal{D}}$-modules from their solutions, following Prosmans-Schneiders' aforementioned work [45]. This requires to pass from the pro-étale site back to the analytic space X. Here one runs into problems: there is a canonical morphism $\chi\colon \mathcal{O} \to \mathrm{R}\,\nu_* \,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$, but we cannot expect it to be an isomorphism, cf. [54, Proposition 6.16(ii)]. The issue boils down to the non-vanishing of the first continuous Galois cohomology of B_{dR}^+. As a solution, Fontaine [30] introduced B_{pdR}^+, the *positive almost de Rham period ring* [2]. We conjecture the existence of an overconvergent version $B_{\mathrm{pdR}}^\dagger$. A corresponding period structure sheaf $\mathcal{O}\mathbb{B}_{\mathrm{pdR}}^\dagger$ should be a suitable bimodule object, giving rise to a functor

$$\mathrm{Rec}\colon\ \mathbf{D}\left(\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right) \to \mathbf{D}\left(\widehat{\mathcal{D}}\right)^{\mathrm{op}}, \mathcal{F}^\bullet \mapsto \mathrm{R}\,\nu_*\,\mathrm{R}\,\underline{\mathcal{H}om}_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\mathcal{F}^\bullet, \mathcal{O}\mathbb{B}_{\mathrm{pdR}}^{\dagger,+}\right).$$

We then expect Rec to be a quasi-left-inverse of Sol, at least on a suitable full subcategory of the derived category of $\widehat{\mathcal{D}}$-modules. In particular, the canonical map

$$\widehat{\mathcal{D}} \longrightarrow \mathrm{Rec}\left(\mathrm{Sol}\left(\widehat{\mathcal{D}}\right)\right) = \mathrm{R}\,\nu_*\,\mathrm{R}\,\underline{\mathcal{H}om}_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}, \mathcal{O}\mathbb{B}_{\mathrm{pdR}}^{\dagger,+}\right) \tag{1.3.1}$$

should be an isomorphism. This would explain how to reconstruct $\widehat{\mathcal{D}}$ from p-adic Hodge theory, thereby justifying the title of this thesis.

1.4 Functional analysis

Ishimura [39] proved that an archimedean analog of (1.3.1) is an isomorphism, provided one takes track of the topologies on the sheaves. This result is a key input for Prosmans-Schneiders' [45]. Consequently, both *loc. cit.* and this thesis have to operate within a framework that accommodates homological algebra whilst accounting for the topologies involved. Following Prosmans-Schneiders' article, we work with ind-objects in the category of k-Banach spaces.

Condensed mathematics [56] offers an alternative approach. It relates to the formalism used in this thesis as follows: The [23, (proof of) Lemma A.15, Proposition A.25], and [40, Proposition 6.1.9] provide a strongly monoidal cocontinuous exact functor into the category of solid k-vector spaces

$$\mathbf{IndBan}_k \to \mathbf{Vec}_k^{\mathrm{solid}}.$$

It translates the results of this thesis into the formalism of $\mathbf{Vec}_k^{\mathrm{solid}}$-valued sheaves.

[2] The acronym pdR comes from the french *presque de Rham.*

1.5 Summary of the thesis

Chapter 2 concerns foundational notions from functional analysis. We introduce various period sheaves and period structure sheaves in chapter 3 and compute their sections explicitly. $\widehat{\mathcal{D}}$ enters the picture in chapter 4, where we construct the bimodule structure on the overconvergent de Rham period structure sheaf and prove the Poincaré Lemma. In chapter 5, we study the positive solution and de Rham functors.

1.6 Conventions and notation

$=$ denotes an equality, $\cong$ denotes an isomorphism, and $\simeq$ denotes an equivalence. When an equality, isomorphism, or equivalence follows from a specific result, the reference is written above the symbol denoting the equality, isomorphism, or equivalence. For example, $X \overset{2.5.10}{\cong} Y$ means: Lemma 2.5.10 implies that X and Y are isomorphic.

Let $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ be an additive functor between two additive categories. It extends to a functor between the associated categories of chain complexes and cochain complexes. Abusing notation, we denote it again by $\mathbf{F}$.

Given a symmetric monoidal category containing a monoid object S, $\mathbf{Mod}(S)$ denotes the category of S-module objects. It is again symmetric monoidal if S is commutative, cf. [18, section 2.2]. Throughout this article, the term *module* always means *left module*, unless explicitly stated otherwise.

The natural numbers are $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. For any natural number n, write $\mathbb{N}_{\geq n} := \{n, n+1, n+2, n+3 \dots\}$.

All filtrations are descending.

Fix a prime number p throughout this article.

All Huber pairs (A, A^+) are complete, that is both A and A^+ are complete as topological rings.

Chapter 2

Functional analysis

We assume that the reader is comfortable with Schneiders' formalism of *quasi-abelian categories*, see [52]. Some of our cited sources operate within the Archimedean setting; nonetheless, the proofs remain applicable in the non-Archimedean context.

2.1 Seminormed, normed, and Banach modules

We follow [13, Chapter 5].

Rings and modules

Definition 2.1.1. A *(non-Archimedean) seminormed ring* is a unitial commutative ring R equipped with a map $|\cdot|\colon R \to \mathbb{R}_{\geq 0}$ such that

- $|0| = 0$,
- $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in R$, and
- there is a $C > 0$ such that $|rs| \leq C|r||s|$ for all $r, s \in R$.

R is a *(non-Archimedean) normed ring* if the following implication holds for all $r \in R$: $|r| = 0$ implies $r = 0$. A normed ring is a *(non-Archimedean) Banach ring* if it is a complete metric space with respect to the metric $(r, s) \mapsto |r - s|$.

The following construction supplies examples of seminormed rings.

Definition 2.1.2. R denotes an abstract commutative ring and $I \subseteq R$ an ideal. Define the *I-adic seminorm on R (with base p)*: Set $|r| := p^{-v}$ for every $r \in R$, where $v \in \mathbb{N} \cup \{\infty\}$ is maximal with respect to the property that $r \in I^v$. Here $I^\infty := \bigcap_{j=0}^{\infty} I^j$ and $p^{-\infty} := 0$. This turns R into a seminormed ring. It is a normed ring when R is separated with respect to the I-adic topology. R is a Banach ring if it is separated and complete with respect to the I-adic topology.

For any $s \in R$, the *s-adic seminorm* is the (s)-adic seminorm.

Lemma 2.1.3. *Consider a map $\phi\colon R \to S$ between two Banach rings that is a morphism of abstract rings, R carries an I-adic norm and S carries a J-adic norm, and $\phi(I) \subseteq J$. Then $|\phi(r)| \leq |r|$ for all $r \in R$.*

Proof. Fix the notation from Definition 2.1.2. Consider $r \in R$ with $|r| = p^{-v}$. Then $r \in I^v$, and $\phi(I) \subseteq J$ implies $\phi(r) \in J^v$. That is $|\phi(r)| \leq p^{-v} = |r|$. □

Definition 2.1.4. Fix a seminormed ring R. A *(non-Archimedean) seminormed R-module* is an R-module M equipped with a map $\|\cdot\|\colon M \to \mathbb{R}_{\geq 0}$ such that

- $\|m + n\| \leq \max\{\|m\|, \|n\|\}$ for all $m, n \in M$ and
- there is a $C > 0$ such that $\|rm\| \leq C|r|\|m\|$ for all $r \in R$, $m \in M$.

M is a *(non-Archimedean) normed R-module* if the following implication holds for all $m \in M$: $|m| = 0$ implies $m = 0$. A normed R-module is a *(non-Archimedean) Banach R-module* if it is a complete metric space with respect to the metric $(m, n) \mapsto |m - n|$.

Categories

Fix a seminormed ring R. We define the categories

$$\mathbf{Ban}_R \subseteq \mathbf{Nrm}_R \subseteq \mathbf{SNrm}_R \tag{2.1.1}$$

of seminormed R-modules, normed R-modules, and R-Banach modules. The morphisms are the R-linear maps $\phi\colon M \to N$ which are *bounded*, that is

$$\|\phi(m)\| \leq C\|m\|$$

for a constant $C = C(\phi) > 0$ and every $m \in M$.

Proposition 2.1.5. *The categories $\mathbf{Ban}_R$, $\mathbf{Nrm}_R$, and $\mathbf{SNrm}_R$ are quasi-abelian. They admit enough functorial projectives.*

Proof. See [13, Propositions 5.1.5 and 5.1.10]. □

We the cite the following two Lemma from [9, Proposition 3.14]. *Loc. cit.* assumes that R is a Banach ring, but the arguments apply as well for seminormed rings.

Lemma 2.1.6. *Let $f\colon M \to N$ be a morphism of R-Banach modules. Then*

(i) $\ker(f) = f^{-1}(0)$ *with the restriction of the norm on M,*

(ii) $\operatorname{coker}(f) = N/\overline{f(M)}$ *with the residue norm,*

(iii) $\operatorname{im}(f) = \overline{f(N)}$ *with the restriction of the norm on N, and*

(iv) $\operatorname{coim}(f) = M/\ker(f)$ *with the residue norm.*

Lemma 2.1.7. *Let $f\colon M \to N$ be a morphism of R-Banach modules.*

(i) It is a monomorphism if and only if it is injective.

(ii) It is an epimorphism if and only if $f(M) \subseteq N$ is dense.

(iii) It is a strict monomorphism if and only if it is injective, the norm on M is equivalent to the norm induced by N, and $f(M)$ is a closed subset of N.

(iv) It is a strict epimorphism if and only if it is surjective and the residue norm on $M/\ker(f)$ is equivalent to the norm on N.

The inclusions (2.1.1) admit left adjoints: the separation and completion functors.

Definition 2.1.8.

(i) The *separation functor* $\mathbf{SNrm}_R \to \mathbf{Nrm}_R$ is $M \mapsto M^{\mathrm{sep}} := M/\overline{\{0\}}$, equipped with the quotient norm. That is, the norm of an element $n \in M^{\mathrm{sep}}$ is $\inf_m \|m\|$, where the infinum runs over all preimages $m \in M$ of n.

(ii) The completion functor $\mathbf{Nrm}_R \to \mathbf{Ban}_R$ sends a seminormed R-module N to its completion $\widehat{N}$; see for example [22, section 1.1.7, the proof of Proposition 5].

(iii) The *separated completion functor* $\mathbf{SNrm}_R \to \mathbf{Ban}_R$ is the composition of the completion functor and the separation functor. Abusing notation, we denote it by $M \mapsto \widehat{M} := \widehat{M^{\mathrm{sep}}}$.

Lemma 2.1.9. *The separated completion functoris exact.*

Proof. See [13, Remark 5.1.7] □

Closed symmetric monoidal structures

R continuous to denote a seminormed ring. Given seminormed R-modules M and N, equip $M \otimes_R N$ with the seminorm

$$\|x\| := \inf \left\{ \max_{i=1,\dots,n} \|m_i\|\|n_i\| \colon x = \sum_{i=1}^{n} m_i \otimes_R n_i \right\}$$

for all $x \in M \otimes_R N$. This defines a bifunctor

$$\mathbf{SNrm}_R \times \mathbf{SNrm}_R \to \mathbf{SNrm}_R, (M, N) \mapsto M \otimes_R N.$$

The *separated tensor product* is

$$\mathbf{Nrm}_R \times \mathbf{Nrm}_R \to \mathbf{Nrm}_R, (M, N) \mapsto M \otimes_R^{\mathrm{sep}} N := (M \otimes_R N)^{\mathrm{sep}},$$

and the *completed tensor product* is

$$\mathbf{Ban}_R \times \mathbf{Ban}_R \to \mathbf{Ban}_R, (M, N) \mapsto M \widehat{\otimes}_R N := \widehat{(M \otimes_R N)}.$$

Lemma 2.1.10. $(\mathbf{SNrm}_R, R, \otimes_R)$*,* $(\mathbf{Nrm}_R, R, \otimes_R^{\mathrm{sep}})$*, and* $\left(\mathbf{Ban}_R, R, \widehat{\otimes}_R\right)$ *are closed symmetric monoidal categories.*

Proof. See the discussion in [13, subsubsection 5.1.1.2], especially *loc. cit.* Corollary 5.1.15. □

Definition 2.1.11. For any two seminormed R-modules M and N, define the *internal homomorphisms* $\underline{\mathrm{Hom}}_R(M, N)$ to be the seminormed R-module of all R-linear bounded functions $\phi \colon M \to N$, together with the seminorm

$$\|\phi\| := \sup_{\substack{m \in M \\ \|m\| \neq 0}} \frac{\|\phi(m)\|}{\|m\|}.$$

Lemma 2.1.12. *Fix* $M \in \mathbf{SNrm}_R$*,* $\mathbf{Nrm}_R$*, or* $\mathbf{Ban}_R$*. Then the assignment* $N \mapsto \underline{\mathrm{Hom}}_R(M, N)$ *defines a right adjoint of the functors* $- \otimes_R M$*,* $- \otimes_R^{\mathrm{sep}} M$*, or* $- \widehat{\otimes}_R M$*, respectively.*

Proof. See the [13, remark following Corollary 5.1.15]. □

Notation 2.1.13. An *R-Banach algebra* S is a possibly non-commutative monoid object in $\mathbf{Ban}_R$. An *S-Banach module* is a left S-module object. $\mathbf{Ban}_S := \mathbf{Mod}(S)$ is the *category of S-Banach modules*, cf. subsection 1.6.

2.2 Ind-Banach modules

Fix a Banach ring R. $\mathbf{Ban}_R$ is neither complete nor cocomplete. Therefore, we consider its ind-completion, cf. [40, chapter 6]. Note that $\widehat{\otimes}_R$ extends to a bifunctor

$$\widehat{\otimes}_R\colon \mathbf{Ind}\left(\mathbf{Ban}_R\right) \times \mathbf{Ind}\left(\mathbf{Ban}_R\right) \to \mathbf{Ind}\left(\mathbf{Ban}_R\right)$$
$$\left(\text{“}\varinjlim_{i\in I}\text{”} V_i\right) \widehat{\otimes}_R \left(\text{“}\varinjlim_{j\in J}\text{”} W_j\right) := \text{“}\varinjlim_{\substack{i\in I\\ j\in J}}\text{”} V_i \widehat{\otimes}_R W_j.$$

Lemma 2.2.1. $\left(\mathbf{Ind}\left(\mathbf{Ban}_R\right), R, \widehat{\otimes}_R\right)$ *is a closed symmetric monoidal elementary quasi-abelian category. It has enough flat projectives stable under the monoidal structure* $\widehat{\otimes}_R$*. Furthermore, it has all limits and colimits.*

Proof. The first sentence follows from Lemma 2.1.10 and [52, Proposition 2.1.19]. The second sentence follows again from [52, Proposition 2.1.19], which applies by the [13, Propositions 5.1.16 and 5.1.17]. The last sentence follows from [40, Proposition 6.1.18] because $\mathbf{Ban}_R$ has finite limits, cf. Lemma 2.1.10 and [59, Tag 002O]. □

Corollary 2.2.2. *Filtered colimits in* $\mathbf{Ind}\left(\mathbf{Ban}_R\right)$ *are strongly exact.*

Proof. This follows from Lemma 2.2.1, by [52, Proposition 2.1.16]. □

Notation 2.2.3. An *R-ind-Banach algebra S* [1] is a possibly non-commutative monoid object in $\mathbf{Ind}\left(\mathbf{Ban}_R\right)$. An *$S$-ind-Banach module* is a left S-module object. $\mathbf{IndBan}_S := \mathbf{Mod}\left(S\right)$ is the *category of S-ind-Banach modules*, cf. subsection 1.6. In particular, $\mathbf{IndBan}_R = \mathbf{Ind}\left(\mathbf{Ban}_R\right)$ is the category of *R-ind-Banach modules.*

Lemma 2.2.4. *Fix a category* $\mathbf{C}$ *and consider the canonical functor* $\mathbf{C} \to \mathbf{Ind}\left(\mathbf{C}\right)$*. It commutes with finite colimits. If* $\mathbf{C}$ *has all finite limits, then* $\mathbf{Ind}\left(\mathbf{C}\right)$ *has all finite limits as well and the canonical functor commutes with those.*

Proof. This is [40, Corollary 6.1.6 and 6.1.17]. □

Lemma 2.2.5. *Let* $\mathbf{E}$ *be a quasi-abelian category and* $f\colon M \to N$ *a morphism in* $\mathbf{Ind}(\mathbf{E})$*. Then* f *is* $\mathbf{P}$ *if and only if* $f = \text{“}\varinjlim_i\text{”} f_i$ *where each* f_i *is* $\mathbf{P}$*. Here,*

$$\mathbf{P} \in \{\textit{mono, epi, strict, strict mono, strict epi}\}.$$

Proof. This is [9, Proposition 2.10]. Its proof implicitly uses Lemma 2.2.4. □

[1] We decide against the usage of the term *ind-R-Banach algebra*. If spoken out loud, it might be misunderstood as an ind-(R-Banach algebra).

Corollary 2.2.6. *Let F denote a field, complete with respect to a non-trivial non-Archimedean valuation. Then, for any F-ind-Banach module V, $-\widehat{\otimes}_F V\colon \mathbf{IndBan}_F \to \mathbf{IndBan}_F$ is exact.*

Proof. Applying Lemma 2.2.4 and 2.2.5, we may assume that V is a k-Banach module. Thus the Corollary follows from [10, Theorem 3.50]. □

Finally, we consider the functor $M \mapsto |M|$ that sends an R-Banach module to its underlying abstract R-module. It extends to a functor

$$\mathbf{IndBan}_R \to \mathbf{Mod}(|R|),\ \text{“}\varinjlim_i\text{”} M_i \mapsto \varinjlim_i |M_i| \tag{2.2.1}$$

Lemma 2.2.7. *The functor (2.2.1) commutes with finite limits.*

Proof. Filtered colimits commute with finite limits by Corollary 2.2.2, thus it suffices to show that the functor $\mathbf{Ban}_R \to \mathbf{Mod}(R)$, $M \mapsto |M|$ commutes with finite limits. In fact, it suffices to compute that it commutes with finite products and equalisers, which is clear. □

Fix a field F, complete with respect to a non-trivial non-Archimedean valuation.

Notation 2.2.8. An *F-Banach space* is an F-Banach module.

Definition 2.2.9. An F-ind-Banach space is *bornological* if it is isomorphic to an object “$\varinjlim$”$_i E_i$ where all the structural maps $E_i \to E_j$ are injective. A *complete bornological F-vector space* is a bornological F-ind-Banach space. $\mathbf{CBorn}_F$ denotes the full subcategory of $\mathbf{IndBan}_F$ of complete bornological F-vector spaces.

Notation 2.2.10. Consider a diagram $i \mapsto E_i$ of complete bornological F-vector spaces. Denote its limit in $\mathbf{CBorn}_F$, if it exists, by $\varprojlim^{\mathrm{b}}_i E_i$. $\varprojlim_i E_i$ is its limit in $\mathbf{IndBan}_F$.

Lemma 2.2.11. *Given a diagram $i \mapsto E_i$ of complete bornological F-vector spaces, $\varprojlim^{\mathrm{b}}_i E_i$ exists and coincides with $\varprojlim_i E_i$.*

Proof. This follows from [9, Remark 3.44 and Proposition 3.60]. □

Lemma 2.2.12. *The functor $\mathbf{CBorn}_F \to \mathbf{IndBan}_F$ is exact.*

Proof. By Corollary [52, Corollary 1.2.28], it suffices to check that the induced functor

$$\mathbf{LH}\,(\mathbf{CBorn}_F) \to \mathbf{LH}\,(\mathbf{IndBan}_F)$$

between the left hearts is exact, cf. *loc. cit.* Definition 1.2.18. The result follows from [46, Proposition 5.16(b)]. We remark that this reference operates in the archimedean context, but the proof goes through in our setting as well. □

$\mathbf{CBorn}_F$ carries a symmetric monoidal operation which we denote by $\widehat{\otimes}^{\mathrm{b}}_F$. In this thesis, we do not need its precise [9, Definition 3.57] but only the following result.

Lemma 2.2.13. *Consider two complete bornological F-vector spaces V and W, which are inverse limits of Banach spaces. Then there is a functorial isomorphism*

$$V\widehat{\otimes}_F W \xrightarrow{\cong} V\widehat{\otimes}^{\mathrm{b}}_F W$$

of F-ind-Banach spaces.

Proof. Both V and W are *proper* as bornological spaces, cf. [9, Definition 3.62]. This follows from [10, Proposition 3.11], together with the fact that Banach spaces are proper, which follows directly from the definition. Lemma 2.2.13 thus follows from [9, Proposition 3.64]. □

Lemma 2.2.14. $-\widehat{\otimes}_F \varprojlim_r F\left\langle \pi^r\zeta_1, \ldots, \pi^r\zeta_d\right\rangle : \mathbf{IndBan}_F \to \mathbf{IndBan}_F$ *is strongly exact, given formal variables $\zeta_1, \ldots, \zeta_d$.*

Proof. It preserves cokernels because the monoidal category $\mathbf{IndBan}_k$ is closed. To show that it preserves kernels of arbitrary maps, apply [9, Remark 2.3] and Corollary 2.2.2; thus it suffices to check that it preserves kernels of maps between k-Banach spaces. Given a k-Banach space V, we compute with the Lemma 2.2.11 and 2.2.13:

$$V\widehat{\otimes}_F \varprojlim_r F\left\langle \pi^r\zeta_1, \ldots, \pi^r\zeta_d\right\rangle \cong V\widehat{\otimes}^{\mathrm{b}}_F \varprojlim_r{}^{\mathrm{b}} F\left\langle \pi^r\zeta_1, \ldots, \pi^r\zeta_d\right\rangle .$$

As $\mathbf{CBorn}_F \hookrightarrow \mathbf{IndBan}_F$ preserves kernels, [18, Corollary 3.51] gives the result. □

2.3 Localisations

Fix a seminormed ring R and an element $r \in R$.

Notation 2.3.1. Given a seminormed R-module M, equip $M[1/r]$ with the seminorm $\|n\| := \inf \|m\|/|r^i|$. The infinum varies along expressions $n = m/r^i$ with $m \in M$.

Lemma 2.3.2. $M \otimes_R R[1/r] \xrightarrow{\cong} M[1/r]$ *as seminormed $R[1/r]$-modules for any R-Banach module M.*

Proof. Denote the canonical map $M \to M[1/r]$ by ϕ. We aim to exploit Yoneda's Lemma to show that it is an isomorphism: We claim

$$\mathrm{Hom}_{R[1/r]}\left(M[1/r], V\right) \to \mathrm{Hom}_R\left(M, V\right), f \mapsto f \circ \phi$$

is a bijection for every seminormed $R[1/r]$-module V. This is clear when we consider the Hom in the category of abstract modules. It remains to check that the Hom also coincide when they only capture the bounded linear maps. That is, we have to check the following for every $R[1/r]$-linear map $f\colon M[1/r] \to V$: f is bounded if and only if $f \circ \phi$ is bounded.

The implication $\Rightarrow$ follows because ϕ is bounded. To prove the converse, fix $n \in M[1/r]$ and a presentation $n = m/r^i$. $C > 0$ is a bound on the scalar multiplication $R[1/r] \times V \to V$. Then

$$\|f(n)\| = \|\frac{(f \circ \phi)(m)}{r^i}\| \le C|\frac{1}{r^i}|\|(f \circ \phi)(m)\| \le C|1|\|f \circ \phi\|\frac{\|m\|}{|r^i|}$$

Taking the infinum over all such expressions $n = m/r^i$, we find $\|f\| \le C|1|\|f \circ \phi\|$. □

Lemma 2.3.3. $-\otimes_R R[1/r]\colon \mathbf{SNrm}_R \to \mathbf{SNrm}_R$ *preserves kernels of maps* $\phi\colon M \to N$ *when* N *is* r*-torsion free.*

Proof. There is a canonical linear map $\tau\colon (\ker\phi)[1/r] \to \ker(\phi[1/r])$, which is bijective. It remains to check that the seminorms on both sides coincide. Recall that for every $x \in (\ker\phi)[1/r]$,

$$\|x\| = \inf_{\substack{x=m/r^i \\ m\in\ker\phi}} \frac{\|m\|}{|r^i|}$$
$$\|\tau(x)\| = \inf_{\substack{x=m/r^i \\ m\in M}} \frac{\|m\|}{|r^i|}.$$

We may therefore compare the infinums' indexing sets

$$S_{\ker\phi} := \left\{\left(m, r^i\right) : x = m/r^i \text{ and } m \in \ker\phi\right\},$$
$$S_M := \left\{\left(m, r^i\right) : x = m/r^i \text{ and } m \in M\right\}.$$

We have to check that both sets coincide. The inclusion $S_{\ker\phi} \subseteq S_M$ is clear. Consider $(m, r^i) \in S_M$ to prove $\supseteq$. Now pick some $\left(\widetilde{m}, r^{\widetilde{i}}\right) \in S_{\ker\phi}$. The equality $m/r^i = x = \widetilde{m}/r^{\widetilde{i}}$ implies $m/1 \in \ker(\phi[1/r])$. That is $\phi(m)/1 = 0$, thus $\phi(m) \in N$ is killed by some power of r. Because N is r-torsion free, this implies $\phi(m) = 0$. We find $(m, r^i) \in S_{\ker\phi}$, as desired. □

Lemma 2.3.4. *The canonical morphism* $\widehat{M \otimes_R N} \xrightarrow{\cong} \widehat{M}\widehat{\otimes}_R\widehat{N}$ *is an isomorphism for any two normed* R*-modules* M *and* N*.*

Proof. This follows from the adjunctions. □

Corollary 2.3.5. *Consider a strictly exact complex* $M' \xrightarrow{\phi'} M \xrightarrow{\phi} M''$ *of* R*-Banach modules. If* M'' *is* r*-torsion free, then*

$$M'\widehat{\otimes}_R\widehat{R[1/r]} \xrightarrow{\phi'\widehat{\otimes}_R \mathrm{id}} M\widehat{\otimes}_R\widehat{R[1/r]} \xrightarrow{\phi\widehat{\otimes}_R \mathrm{id}} M''\widehat{\otimes}_R\widehat{R[1/r]} \tag{2.3.1}$$

is a strictly exact complex of $\widehat{R[1/r]}$*-Banach modules.*

Proof. By assumption, ϕ' is strict and the canonical morphism $\iota\colon \operatorname{im}\phi' \to \ker\phi$ is an isomorphism. We apply $-\widehat{\otimes}_R R[1/r]$ to ι:

- Regarding its domain,

$$\begin{aligned}(\operatorname{im}\phi')\otimes_R R[1/r] &= \ker(M \to \operatorname{coker}\phi')\otimes_R R[1/r]\\ &\overset{2.3.3}{\cong} \ker(M\otimes_R R[1/r] \to \operatorname{coker}(\phi')\otimes_R R[1/r])\\ &\cong \ker\left(M\otimes_R R[1/r] \to \operatorname{coker}\left(\phi'\otimes_R \mathrm{id}_{R[1/r]}\right)\right)\\ &= \operatorname{im}\left(\phi'\otimes_R \mathrm{id}_{R[1/r]}\right).\end{aligned}$$

  The application of Lemma 2.3.3 requires that the canonical morphism $M \to \operatorname{coker}\phi'$ is strict and $\operatorname{coker}\phi'$ does not have r-torsion. The strictness is formal, see [52, Remark 1.1.2(a)]. Regarding the torsion, consider $[m] \in \operatorname{coker}\phi' = M/\ker\phi$ such that $r[m] = 0$. That is $rm \in \ker\phi$. This implies $r\phi(m) = \phi(rm) = 0$, thus $\phi(m) = 0$ because M'' is r-torsion free. Therefore, $[m] = 0$.

- Regarding its codomain,

$$(\ker\phi)\otimes_R R[1/r] \cong \ker\left(\phi\otimes_R \mathrm{id}_{R[1/r]}\right)$$

  by Lemma 2.3.3.

It follows that $\iota\otimes_R\mathrm{id}_{R[1/r]}$ coincides with the canonical morphism $\operatorname{im}\left(\phi'\otimes_R\mathrm{id}_{R[1/r]}\right) \to \ker\left(\phi\otimes_R\mathrm{id}_{R[1/r]}\right)$, which is thus an isomorphism. That is

$$M'\otimes_R R[1/r] \xrightarrow{\phi'\otimes_R \mathrm{id}} M\otimes_R R[1/r] \xrightarrow{\phi\widehat{\otimes}_R \mathrm{id}} M''\otimes_R R[1/r]$$

is strictly exact. Now apply the separated completion functor; the Lemmata 2.3.4 and 2.1.9 imply that (2.3.2) is strictly exact. ☐

Let F be a field which is complete with respect to a non-trivial non-Archimedean valuation. Fix a *pseudo-uniformiser* $\pi \in F$, that is $0 < |\pi| < 1$. The Banach ring of *power-bounded elements* is $F^\circ = \{x \in F\colon |x| \le 1\}$. Consider $R = F^\circ$ and $r = \pi$.

Lemma 2.3.6. *Let M denote an F°-Banach module carrying the π-adic norm. Assume further that M has no π-torsion. Then $M[1/\pi]$ is an F-Banach space, in particular $M\widehat{\otimes}_{F^\circ}F \xrightarrow{\cong} M[1/\pi]$.*

Proof of Lemma 2.3.6. M is the unit ball of $M[1/\pi]$ because it is π-torsion free. Any given Cauchy sequence in $M[1/\pi]$ is bounded, thus we can assume it lies in M. Since M is complete, it follows that such a sequence converges in M, therefore it converges in $M[1/\pi]$. This shows that $M[1/\pi]$ is complete. Finally, apply Lemma 2.3.2. □

Corollary 2.3.7. *Consider a strictly exact complex $M' \xrightarrow{\phi'} M \xrightarrow{\phi} M''$ of F°-Banach modules. If M'' is π-torsion free, then*

$$M'\widehat{\otimes}_{F^\circ}F \xrightarrow{\phi'\widehat{\otimes}_R \mathrm{id}} M\widehat{\otimes}_{F^\circ}F \xrightarrow{\phi\widehat{\otimes}_R \mathrm{id}} M''\widehat{\otimes}_{F^\circ}F \tag{2.3.2}$$

is a strictly exact complex of F-Banach spaces.

Proof. $\widehat{F^\circ[1/p]} \xrightarrow{\cong} F$ by Lemma 2.3.6, thus Corollary 2.3.5 applies. □

2.4 Banach and ind-Banach modules of power series

Fix a Banach ring R.

Notation 2.4.1. Let Ω denote a possibly infinite set. $\mathbb{N}^{(\Omega)}$ is the set of tuples $\alpha = (\alpha_\omega)_{\omega\in\Omega} \in \mathbb{N}^\Omega$ such that $|\alpha| := \sum_{\omega\in\Omega}\alpha_\omega$ is finite. Write

$$\zeta^\alpha := \prod_{\omega\in\Omega}\zeta_\omega^{\alpha_\omega}$$

for a given set of formal variables $\{\zeta_\omega : \omega\in\Omega\}$ and all $\alpha = (\alpha_\omega)_{\omega\in\Omega} \in \mathbb{N}^{(\Omega)}$.

Definition 2.4.2. Let M denote an R-Banach module and Ω a fixed set. An element of the R-module $M\langle\zeta_\omega : \omega\in\Omega\rangle$ is a formal expression

$$\sum_{\alpha\in\mathbb{N}^{(\Omega)}} m_\alpha\zeta^\alpha \in \prod_{\alpha\in\mathbb{N}^{(\Omega)}} M\zeta^\alpha,$$

such that the sets $\left\{\alpha\in\mathbb{N}^{(\Omega)} : m_\alpha \geq \epsilon\right\}$ are finite for all $\epsilon > 0$. We equip $M\langle\zeta_\omega : \omega\in\Omega\rangle$ with the norm

$$\|\sum_{\alpha\in\mathbb{N}^{(\Omega)}} m_\alpha\zeta^\alpha\| := \sup_{\alpha\in\mathbb{N}^{(\Omega)}}\|m_\alpha\|.$$

Remark 2.4.3. When $R = F$ is a field, complete with respect to a non-trivial non-Archimedean valuation, there is the canonical isomorphism

$$c_0\left(\mathbb{N}^{(\Omega)}\right) \xrightarrow{\cong} M\left\langle \zeta_\omega \colon \omega \in \Omega \right\rangle,$$
$$\left(\phi\colon \mathbb{N}^{(\Omega)} \to F\right) \mapsto \sum_{\alpha \in \mathbb{N}^{(\Omega)}} \phi\left(\alpha\right) \zeta^\alpha,$$

cf. [47, chapter 3]. We chose the notation $M\left\langle \zeta_\omega \colon \omega \in \Omega \right\rangle$ to emphasise that Definition 2.4.2 is a generalisaton of Tate algebras, in view of section 2.5.

Notation 2.4.4. $\mathbf{Ban}_R^{\leq 1}$ denotes the category whose objects are R-Banach modules and whose morphisms are the R-linear maps which are bounded by 1.

Let $\{M_i\}_{i \in I}$ denote a family of R-Banach modules. Their coproduct exists in $\mathbf{Ban}_R^{\leq 1}$, cf. [13, subsubsection 5.1.1.1]. Denote it by $\coprod_{i\in I}^{\leq 1} M_i$. It is the *non-expanding coproduct.*

Lemma 2.4.5. *Let M denote an R-Banach module and Ω a fixed set. Then the universal property of the non-expanding coproduct induces an isomorphism*

$$\coprod_{\alpha \in \mathbb{N}^{(\Omega)}}^{\leq 1} M\zeta^\alpha \xrightarrow{\cong} M\left\langle \zeta_\omega \colon \omega \in \Omega \right\rangle \tag{2.4.1}$$

of R-Banach modules.

Proof. [13, Subsubsection 5.1.1.1] explains that $\coprod_{\alpha \in \mathbb{N}^{(\Omega)}}^{\leq 1} M\zeta^\alpha$ is the completion of the direct sum $\bigoplus_{\alpha \in \mathbb{N}^{(\Omega)}} M\zeta^\alpha$ equipped with the norm

$$\|\left(m_\alpha \zeta^\alpha\right)_{\alpha \in \mathbb{N}^{(\Omega)}}\| := \sup_{\alpha \in \mathbb{N}^{(\Omega)}} \|m_\alpha\|.$$

On the other hand, $M\left\langle \zeta_\omega \colon \omega \in \Omega \right\rangle$ is the completion of $M\left[\zeta_\omega \colon \omega \in \Omega\right]$, the subspace of all finite sums $\sum_{\alpha \in \mathbb{N}^{(\Omega)}} m_\alpha \zeta^\alpha$ carrying the induced norm

$$\|\sum_{\alpha \in \mathbb{N}^{(\Omega)}} m_\alpha \zeta^\alpha\| = \sup_{\alpha \in \mathbb{N}^{(\Omega)}} \|m_\alpha\|.$$

The morphism (2.4.1) thus arises as the completion of the isomorphism

$$\bigoplus_{\alpha \in \mathbb{N}^{(\Omega)}} M\zeta^\alpha \xrightarrow{\cong} M\left[\zeta_\omega \colon \omega \in \Omega\right]$$

of normed R-modules. □

Lemma 2.4.6. *M is an R-Banach module and Ω a fixed set. Given any $m := (m_\alpha)_{\alpha\in\mathbb{N}^{(\Omega)}} \subseteq M$ with $\|rm_\alpha\| = |r|\|m_\alpha\|$ for all $r \in R$ and $\|m_\alpha\| \leq 1$ for all $\alpha \in \mathbb{N}^{(\Omega)}$,*

$$\sum_{\alpha\in\mathbb{N}^{(\Omega)}} r_\alpha \zeta^\alpha \mapsto \sum_{\alpha\in\mathbb{N}^{(\Omega)}} r_\alpha m_\alpha$$

defines a morphism $R\langle\zeta_\omega\colon \omega\in\Omega\rangle \to M$ of R-Banach modules.

Proof. By Lemma 2.4.5, we can exploit the universal property of the non-expanding coproduct. Thus we may check that the maps $R \to T$, $r \mapsto rm_\alpha$ are bounded by 1 for all $\alpha \in \mathbb{N}^\Omega$: for all $r \in R$, $\|rm_\alpha\| \leq |r|\|m_\alpha\| \leq |r|\cdot 1$. □

Lemma 2.4.5(i) implies that the operation $M \mapsto M\langle\zeta_\omega\colon \omega\in\Omega\rangle$ defines a functor $\mathbf{Ban}_R \to \mathbf{Ban}_R$. By [40, Proposition 6.1.9], it extends canonically to a functor $\mathbf{IndBan}_R \to \mathbf{IndBan}_R$, which we denote again by $M_\bullet \mapsto M_\bullet\langle\zeta_\omega\colon \omega\in\Omega\rangle$. *Loc. cit.* explains that this is not an abuse of notation, as the canonical morphisms

$$\text{“}\varinjlim_{i\in I}\text{”}\left(M_i\langle\zeta_\omega\colon \omega\in\Omega\rangle\right) \xrightarrow{\cong} M_\bullet\langle\zeta_\omega\colon \sigma\in\Omega\rangle$$

are isomorphisms for all $M_\bullet = \text{“}\varinjlim\text{”}_{i\in I} M_i \in \mathbf{IndBan}_R$.

Lemma 2.4.7. *Fix an R-ind-Banach module $M_\bullet = \text{“}\varinjlim\text{”}_{i\in I} M_i$ and a set Ω. Then*

$$M_\bullet \widehat{\otimes}_R R\langle\zeta_\omega\colon \omega\in\Omega\rangle \xrightarrow{\cong} M_\bullet\langle\zeta_\omega\colon \omega\in\Omega\rangle, \tag{2.4.2}$$

the canonical morphism, is an isomorphism.

Proof. We may assume without loss of generality that $M_\bullet = M$ is an R-Banach module. The Lemma then follows from Lemma 2.4.5 and the [13, last sentence in the proof of Proposition 5.1.16]. □

Corollary 2.4.8. *$M_\bullet \mapsto M_\bullet\langle\zeta_\omega\colon \omega\in\Omega\rangle$ is strongly exact for every set Ω.*

Proof. This follows from Lemma 2.4.7 and [13, Proposition 5.1.16]. □

Notation 2.4.9. $\beta+\gamma := (\beta_\omega+\gamma_\omega)_{\omega\in\Omega} \in \mathbb{N}^{(\Omega)}$ for $\beta = (\alpha_\omega)_{\omega\in\Omega}, \gamma = (\gamma_\omega)_{\omega\in\Omega} \in \mathbb{N}^{(\Omega)}$.

Lemma 2.4.10. *Let S denote an R-Banach algebra and Ω a fixed set. Then*

$$\left(\sum_{\alpha\in\mathbb{N}^{(\Omega)}} s_{1\alpha}\zeta^\alpha\right)\cdot\left(\sum_{\alpha\in\mathbb{N}^{(\Omega)}} s_{2\alpha}\zeta^\alpha\right) := \sum_{\alpha\in\mathbb{N}^{(\Omega)}}\left(\sum_{\alpha=\beta+\gamma} s_{1\beta}s_{2\gamma}\right)\zeta^\alpha$$

makes $S\langle\zeta_\omega\colon \omega\in\Omega\rangle$ an S-Banach algebra.

Proof. This is obvious. □

Lemma 2.4.11. *Consider a morphism $\phi\colon S \to T$ of R-Banach algebras and a set Ω. Given any tuple $t := (t_\omega)_{\omega\in\Omega} \in T^\Omega$ with $\|rt^\alpha\| = |r|\|t^\alpha\|$ for all $r \in R$ and $\|t^\alpha\| \leq 1$ for all $t \in \mathbb{N}^{(\Omega)}$,*

$$\sum_{\alpha\in\mathbb{N}^{(\Omega)}} s_\alpha \zeta^\alpha \mapsto \sum_{\alpha\in\mathbb{N}^{(\Omega)}} \phi(s_\alpha)\, t^\alpha$$

defines a morphism $S\langle\zeta_\omega\colon \omega\in\Omega\rangle \to T$ of R-Banach algebras.

Proof. See Lemma 2.4.6 for the construction of the morphism

$$R\langle\zeta_\omega\colon \omega\in\Omega\rangle \to T,\ \sum_{\alpha\in\mathbb{N}^{(\Omega)}} r_\alpha \zeta^\alpha \mapsto \sum_{\alpha\in\mathbb{N}^{(\Omega)}} r t^\alpha$$

of R-Banach modules. Apply Lemma 2.4.7 and [52, Proposition 1.5.2] to lift it to the desired morphism. It is bounded, S-linear, and multiplicative by construction. □

Fix a pseudo-uniformiser $\pi \in R$, that is $0 < |\pi| < 1$.

Notation 2.4.12. Let M be an R-Banach module, Ω a fixed set, and $q \in \mathbb{N}$. Define

$$M\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle := M\langle\zeta_\omega\colon \omega\in\Omega\rangle\langle\eta_\omega\colon \omega\in\Omega\rangle/\overline{(\pi^q\eta_\omega - \zeta_\omega\colon \omega\in\Omega)},$$

an R-Banach module, where the η_ω denote formal variables. We write ζ_ω/p^q for the images of the η_ω in $M\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle$.

If $M = S$ is an R-Banach algebra, we view $S\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle$ as an R-Banach algebra where the multiplication is induced by Lemma 2.4.10.

Fix $q \in \mathbb{N}$. The operation $M \mapsto M\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle$ defines a functor $\mathbf{Ban}_R \to \mathbf{Ban}_R$. By [40, Proposition 6.1.9], it extends canonically to a functor $\mathbf{IndBan}_R \to \mathbf{IndBan}_R$, which we denote again by $M_\bullet \mapsto M_\bullet\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle$. *Loc. cit.* explains that this is not an abuse of notation, as the canonical morphisms

$$\text{“}\varinjlim_{i\in I}\text{”}\left(M_i\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle\right) \xrightarrow{\cong} M_\bullet\left\langle\frac{\zeta_\omega}{\pi^q}\colon \sigma\in\Omega\right\rangle$$

are isomorphisms for all $M_\bullet = \text{“}\varinjlim\text{”}_{i\in I} M_i \in \mathbf{IndBan}_R$.

Lemma 2.4.13. *$M_\bullet \mapsto M_\bullet\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle$ is strongly exact for every $q \in \mathbb{N}$ and set Ω.*

Proof. Fix an R-Banach module M and consider the morphism

$$M\langle\eta_\omega\colon \omega\in\Omega\rangle \to M\left\langle\frac{\zeta_\omega}{\pi^q}\colon \omega\in\Omega\right\rangle, \eta_\omega \mapsto \frac{\zeta_\omega}{\pi^q}.$$

It is an isomorphism, as one directly constructs a two-sided inverse via the universal property of the non-expanding coproduct, see Lemma 2.4.5. Lemma 2.4.13 thus follows from Corollary 2.4.8. □

Lemma 2.4.14. *Let M denote an R-Banach module, Ω a fixed set, and $q \in \mathbb{N}$. Then*

$$M \langle \zeta_\omega \colon \omega \in \Omega \rangle \to M \left\langle \frac{\zeta_\omega}{\pi^{q+1}} \colon \omega \in \Omega \right\rangle$$

lifts canonically to a morphism of R-Banach modules as follows:

$$M \left\langle \frac{\zeta_\omega}{\pi^{q}} \colon \omega \in \Omega \right\rangle \to M \left\langle \frac{\zeta_\omega}{\pi^{q+1}} \colon \omega \in \Omega \right\rangle$$

If $M = S$ is an R-Banach algebra, this defines a morphism of R-Banach algebras.

Proof. Exploit the universal property of the non-expanding coproduct, cf. Lemma 2.4.5, to write down the morphism

$$M \langle \zeta_\omega \colon \omega \in \Omega \rangle \langle \eta_\omega \colon \omega \in \Omega \rangle \to M \left\langle \frac{\zeta_\omega}{\pi^{q+1}} \colon \omega \in \Omega \right\rangle,$$

given by $\eta_\omega^\alpha \mapsto \pi^{|\alpha|} \left(\zeta_\omega / \pi^q\right)^\alpha$ for all $\alpha \in \mathbb{N}^{(\Omega)}$. It factors through

$$M \langle \zeta_\omega \colon \omega \in \Omega \rangle \langle \eta_\omega \colon \omega \in \Omega \rangle / \left(\pi^q \eta_\omega - \zeta_\omega \colon \omega \in \Omega\right) \to M \left\langle \frac{\zeta_\omega}{\pi^{q+1}} \colon \omega \in \Omega \right\rangle.$$

Complete to get the desired map. The last sentence of Lemma 2.4.14 is clear. □

Lemma 2.4.14 furnishes a commutative diagram of R-Banach modules:

$$M \langle \zeta_\omega \colon \omega \in \Omega \rangle \longrightarrow M \left\langle \frac{\zeta_\omega}{\pi} \colon \omega \in \Omega \right\rangle \longrightarrow M \left\langle \frac{\zeta_\omega}{\pi^2} \colon \omega \in \Omega \right\rangle \longrightarrow \dots . \tag{2.4.3}$$

Definition 2.4.15. Let M denote an R-Banach module and Ω a fixed set. We denote the formal colimit of the diagram (2.4.3) by

$$M \left\langle \frac{\zeta_\omega}{\pi^\infty} \colon \omega \in \Omega \right\rangle := \text{“}\varinjlim_{q \in \mathbb{N}}\text{”} M \left\langle \frac{\zeta_\omega}{\pi^q} \colon \omega \in \Omega \right\rangle.$$

This is, by definition, an R-ind-Banach module. If $M = S$ is an R-Banach algebra, we view $S \left\langle \frac{\zeta_\omega}{\pi^\infty} \colon \omega \in \Omega \right\rangle$ as an R-ind-Banach algebra.

Following previous discussion, $M \mapsto M \left\langle \frac{\zeta_\omega}{\pi^\infty} \colon \omega \in \Omega \right\rangle$ extends to a functor

$$\mathbf{IndBan}_R \to \mathbf{IndBan}_R, \text{“}\varinjlim_{i \in I}\text{”} M_i \mapsto \varinjlim_{i \in I} M_i \left\langle \frac{\zeta_\omega}{\pi^\infty} \colon \omega \in \Omega \right\rangle.$$

which we denote by $M_\bullet \mapsto M_\bullet \left\langle \frac{\zeta_\omega}{\pi^\infty} \colon \omega \in \Omega \right\rangle$. *Loc. cit.* explains that this is not an abuse of notation, as the canonical morphisms

$$\text{“}\varinjlim_{i \in I}\text{”} \left(M_i \langle \zeta_\omega \colon \omega \in \Omega \rangle\right) \xrightarrow{\cong} M_\bullet \langle \zeta_\omega \colon \sigma \in \Omega \rangle$$

are isomorphisms for all $M_\bullet = \text{“}\varinjlim\text{”}_{i \in I} M_i \in \mathbf{IndBan}_R$.

Lemma 2.4.16. $M_\bullet \mapsto M_\bullet \left\langle \frac{\zeta_\omega}{\pi^\infty} \colon \omega \in \Omega \right\rangle$ *is strongly exact for every set Ω.*

Proof. By Lemma 2.2.5, it suffices to check the lemma for the restriction of the functor to $\mathbf{Ban}_R$. Now the result follows from Lemma 2.4.13 and Corollary 2.2.2. □

2.5 Banach and ind-Banach completions

We continue to fix a Banach ring R, together with a pseudo-uniformiser $\pi \in R$.

Recall Notation 2.4.12.

Definition 2.5.1. Fix a commutative R-Banach algebra S and a subset $\Sigma \subseteq S$. Set

$$S\left\langle \frac{\Sigma}{\pi^q} \right\rangle := S\left\langle \frac{\zeta_\sigma}{\pi^q} : \sigma \in \Sigma \right\rangle \Big/ \overline{\left(\sigma - \pi^q \frac{\zeta_\sigma}{\pi^q} : \sigma \in \Sigma\right)},$$

for all $q \in \mathbb{N}$. This is, by definition, an R-Banach algebra. We write σ/p^q for the images of the ζ_σ/π^q in $S\left\langle \frac{\Sigma}{\pi^q} \right\rangle$.

Notation 2.5.2. When $\Sigma = \{s_1, \ldots, s_d\}$ is finite, abbreviate

$$S\left\langle \frac{s_1, \ldots, s_d}{\pi^q} \right\rangle := S\left\langle \frac{\Sigma}{\pi^q} \right\rangle.$$

Remark 2.5.3. The ideal $\left(\sigma - p^q \frac{\zeta_\sigma}{\pi^q} : \sigma \in \Sigma\right)$ in Definition 2.5.1 is not closed in general, see for example [11, Proposition 5.7].

Lemma 2.5.4. *For a commutative R-Banach algebra S, a subset $\Sigma \subset S$, and $q \in \mathbb{N}$,*

$$\delta^q : S\left\langle \frac{\zeta_\omega}{\pi^q} : \omega \in \Omega \right\rangle \to S\left\langle \frac{\zeta_\omega}{\pi^{q+1}} : \omega \in \Omega \right\rangle$$

denotes the map constructed by Lemma 2.4.14. It factors through a morphism

$$S\left\langle \frac{\Sigma}{\pi^q} \right\rangle \to S\left\langle \frac{\Sigma}{\pi^{q+1}} \right\rangle$$

of R-Banach algebras.

Proof. The computations

$$\delta^q\left(\sigma - \pi^q \frac{\zeta_\sigma}{\pi^q}\right) = \sigma - \pi^{q+1} \frac{\zeta_\sigma}{\pi^{q+1}}$$

for all $\sigma \in \Sigma$ imply

$$\delta^q\left(\left(\sigma - \pi^q \frac{\zeta_\sigma}{\pi^q} : \sigma \in \Sigma\right)\right) \subseteq \left(\sigma - \pi^{q+1} \frac{\zeta_\sigma}{\pi^{q+1}} : \sigma \in \Sigma\right).$$

The δ^q are bounded, thus the statement for the closures of the ideals follows. □

Lemma 2.5.4 gives a commutative diagram

$$S\langle \Sigma \rangle \longrightarrow S\left\langle \frac{\Sigma}{\pi^1} \right\rangle \longrightarrow S\left\langle \frac{\Sigma}{\pi^2} \right\rangle \longrightarrow \ldots. \tag{2.5.1}$$

Definition 2.5.5. Let S denote a commutative R-Banach algebra and $\Sigma \subset S$ a fixed subset. We denote the formal colimit of the diagram (2.4.3) by

$$S\left\langle \frac{\Sigma}{\pi^\infty} \right\rangle := \text{“}\varinjlim_{q\in\mathbb{N}}\text{”} S\left\langle \frac{\Sigma}{\pi^q} \right\rangle.$$

This is, by definition, an R-ind-Banach algebra.

Notation 2.5.6. When $\Sigma = \{s_1, \ldots, s_d\} \subseteq S$ is finite, abbreviate

$$S\left\langle \frac{s_1, \ldots, s_d}{\pi^\infty} \right\rangle := S\left\langle \frac{\Sigma}{\pi^\infty} \right\rangle$$

Definition 2.5.7. An R-Banach module M is *bounded* if

$$\sup_{m\in M} \|m\| < C$$

for some constant $C = C(M) > 0$. In this case, M is *bounded by* C.

Example 2.5.8. $R = F$ is a field, complete with respect to a non-Archimedean non-trivial valuation. $S = F\langle T, L\rangle$ is the Tate algebra in two variables and Σ is its ideal generated by $T - L$. Then the canonical morphism

$$F\langle T, L\rangle \left\langle \frac{T-L}{\pi} \right\rangle \xrightarrow{\not\cong} F\langle T, L\rangle \left\langle \frac{\Sigma}{\pi} \right\rangle$$

is not injective: $T - L$ is a non-zero element of the domain, but it vanishes in $F\langle T, L\rangle \langle \Sigma/\pi\rangle$. Indeed, for all $i \in \mathbb{N}$,

$$|\pi^i|\|T - L\| = \|\pi^i (T - L)\| \leq \|\zeta_{\pi^i(T-L)}\| \leq 1.$$

This implies $\|T - L\| = 0$, thus $T - L = 0$.

We resolve this issue by considering $R = F^\circ$, $S = F^\circ\langle T, L\rangle$, and $\Sigma = (T - L)$ instead. Then $F^\circ\langle T, L\rangle$ is bounded by 1, thus Lemma 2.5.9 applies.

Lemma 2.5.9. *Fix two commutative R-Banach algebras S and T as well as a subset $\Sigma \subseteq S$. We further assume that T is bounded by 1 and for all $t_1, t_2 \in T$, $\|t_1t_2\| \leq \|t_2\|\|t_2\|$. Also, $\pi = \pi \cdot 1 \in T$ is not a zero-divisor. Then, for every $q \in \mathbb{N}$, there is a bijection between the set of morphisms $\phi\colon S\langle \Sigma/\pi^q\rangle \to T$ of R-Banach algebras and the set of morphisms of $\psi\colon S \to T$ of R-Banach algebras such that*

(i) π^q divides $\phi(\sigma) \in T$ for all $\sigma \in \Sigma$ and

(ii) $\|\phi(\sigma)\| \leq \|\pi^q\|$ for all $\sigma \in \Sigma$. Here, $\|\cdot\|$ denotes the norm on T.

The bijection is given by $\phi \mapsto \psi := \phi|_S$.

Proof. Firstly, we show that the assignment $\phi \mapsto \phi|_S$ defines the desired map. Indeed, given a morphism $\phi\colon S\langle\Sigma/\pi^q\rangle \to T$ of R-Banach algebras, we find that

(i) $\phi(\sigma) = \pi^q\phi(\sigma/\pi^q) \in T$. That is, π^q divides $\phi(\sigma)$.

(ii) Furthermore,

$$\|\phi(\sigma)\| \leq \|\pi^q\phi(\sigma/\pi^q)\| \leq \|\pi^q\|\|\phi(\sigma/\pi^q)\| \leq \|\pi^q\|1 = \|\pi^q\|.$$

This computation relies on the assumptions on T.

Secondly, we check that $\phi \mapsto \phi|_S$ is injective. Pick two maps $\phi, \phi'\colon S\left\langle\frac{\Sigma}{\pi^q}\right\rangle \to T$ which agree on S. It suffices to show that they agree on elements of the form σ/π^q where $\sigma \in \Sigma$. But we compute

$$\pi^q\phi\left(\frac{\sigma}{\pi^q}\right) = \phi(\sigma) = \phi'(\sigma) = \pi^q\phi'\left(\frac{\sigma}{\pi^q}\right).$$

Since $\pi \in T$ is not a zero-divisor, we find $\phi(\sigma/\pi^q) = \phi'(\sigma/\pi^q)$.

Thirdly, we prove that $\phi \mapsto \phi|_S$ is surjective. That is, we have to extend a given $\psi\colon S \to T$ satisfying (i) and (ii) to a morphism $S\langle\Sigma/\pi^q\rangle \to T$ of R-Banach algebras. Extend ψ with Corollary 2.4.11 to a morphism

$$S\left\langle\frac{\zeta_\sigma}{\pi^q}\colon \sigma \in \Sigma\right\rangle \to T, \frac{\zeta_\sigma}{\pi^q} \mapsto \frac{\phi(\sigma)}{\pi^q}$$

of R-Banach algebras. It vanishes on the ideal generated by all the $\sigma - \pi^q\zeta_\sigma/\pi^q$, and thus it vanishes on the closure. Therefore, the map above factors through a morphism

$$\phi\colon S\left\langle\frac{\Sigma}{\pi^q}\right\rangle \to T$$

of R-Banach algebras. By construction, $\phi|_S = \psi$. □

Lemma 2.5.10. *Consider a commutative* R*-Banach algebra* S *which is bounded by* 1 *and for all* $s_1, s_2 \in S$, $\|s_1s_2\| \leq \|s_1\|\|s_2\|$. *Fix a subset* $\Sigma \subseteq S$. $I := (\Sigma)$ *is the ideal generated by this subset. Then, for every* $q \in \mathbb{N}$, *the canonical morphism*

$$S\left\langle\frac{\Sigma}{\pi^q}\right\rangle \xrightarrow{\cong} S\left\langle\frac{I}{\pi^q}\right\rangle \tag{2.5.2}$$

is an isomorphism of S*-Banach algebras. It is an isomorphisms of* S*-ind-Banach algebras for* $q = \infty$.

Proof. Assume $q < \infty$ without loss of generality and construct the morphism

$$S\left\langle \frac{\zeta_i}{\pi^q} : i \in I \right\rangle \to S\left\langle \frac{\Sigma}{\pi^q} \right\rangle, \frac{\zeta_i}{\pi^q} \mapsto \frac{i}{\pi^q}$$

of S-Banach algebras with Lemma 2.4.11. It factors through the desired two-sided inverse of (2.5.2). □

Proposition 2.5.11. *Fix a commutative ring S containing a finitely generated ideal I, such that S is I-adically separated and complete. Equip S with the I-adic norm. Let ζ denote a formal variable. Then the topological algebra underlying the Banach algebra $S\langle\zeta\rangle$ carries the (I)-adic topology, where $(I) \subseteq S\langle\zeta\rangle$ is the ideal generated by the image of I in $S\langle\zeta\rangle$.*

Now fix an element $s \in S$ and a natural number q. Then the topological algebra underlying the Banach algebra $S\left\langle \frac{s}{\pi^q} \right\rangle$ carries the (I)-adic topology, where $(I) \subseteq S\langle s/\pi^q\rangle$ is the ideal generated by the image of I in $S\langle s/\pi^q\rangle$.

We need the following two Lemmata in order to prove Proposition 2.5.11.

Lemma 2.5.12. *Fix a surjective map $\phi\colon A \to B$ of abstract commutative rings, together with an ideal $J \subseteq A$ such that A is J-adically separated and complete. Equip B with the quotient topology. Then B carries the $\phi(J)$-adic topology.*

Proof. A subset $U \subseteq B$ is open if its preimage $\phi^{-1}(U) \subseteq A$ is open. In this case, there exists an $n \in \mathbb{N}$ such that $J^n \subseteq \phi^{-1}(U)$, thus $\phi(J)^n = \phi(J^n) \subseteq U$.

It remains to show that for each $n \in \mathbb{N}$, $\phi(J)^n \subseteq B$ is open. Again, this is the case once its preimage is open. But its preimage $\phi^{-1}(\phi(J)^n)$ is an ideal in A and it contains J^n. This implies that it is open. □

Lemma 2.5.13. *Let S commutative R-Banach algebra, $s \in S$, and $q \in \mathbb{N}$. Then*

$$S\langle\zeta\rangle / \overline{(\pi^q\zeta - s)} \xrightarrow{\cong} S\left\langle \frac{s}{\pi^q} \right\rangle, \zeta \mapsto \frac{s}{\pi^q} \tag{2.5.3}$$

is an isomorphism of S-Banach algebras.

Proof. One constructs the map (2.5.3) and its two-sided inverse via Lemma 2.4.11. □

Proof of Proposition 2.5.11. Fix a power series $f = \sum_{\alpha\geq 0} f_\alpha\zeta^\alpha \in S\langle\zeta\rangle$. We have, by definition, $\|f\| \leq p^{-r}$ if and only if $\|f_\alpha\| \leq p^{-r}$ for all $\alpha \geq 0$. In order to prove the first statement, we therefore have to show

$$\|f_\alpha\| \leq p^{-r} \text{ for all } \alpha \geq 0 \Leftrightarrow f \in (I)^r. \tag{2.5.4}$$

The direction $\Leftarrow$ is clear. It remains to check $\Rightarrow$. Write $J := I^r$ and fix a finite generating set $(x_1, \ldots, x_n) = J$. Define $e_\alpha := \sup_{f_\alpha \in J^e} e$ for all $\alpha \geq 0$. Note that for all $\alpha \geq 0$, the assumption $\|f_\alpha\| \leq p^{-r}$ implies $f \in I^r = J$, which gives $e_\alpha \geq 1$. Therefore, we can write, for all $\alpha \geq 0$,

$$f_\alpha = \sum_{i=1}^{n} f_{i\alpha} x_i$$

for certain $f_{i\alpha} \in J^{e_\alpha - 1}$. Also, $f_\alpha \to 0$ for $\alpha \to \infty$ implies

$$e_\alpha \to \infty \text{ for } \alpha \to \infty. \tag{2.5.5}$$

Thus $\|f_{i\alpha}\| \leq p^{-(e_\alpha - 1)} \to 0$ for $\alpha \to \infty$, and the formal power series $f_i := \sum_{\alpha \geq 0} f_{i\alpha} \zeta^\alpha$ define elements of $S\langle\zeta\rangle$ for all $i = 1, \ldots, n$. But then

$$f = \sum_{\alpha \geq 0} f_\alpha \zeta^\alpha = \sum_{\alpha \geq 0} \sum_{i=1}^{n} f_{i\alpha} x_i \zeta^\alpha = \sum_{i=1}^{n} \left(\sum_{\alpha \geq 0} f_{i\alpha} \zeta^\alpha \right) x_i = \sum_{i=1}^{n} f_i x_i \in (J) = (I)^r.$$

This finishes the proof of $\Rightarrow$ in (2.5.4), and we get the first half of Proposition 2.5.11. Apply Lemma 2.5.12 to $S\langle\zeta\rangle \to S\langle\zeta\rangle / \overline{(\pi^q \zeta - s)} \overset{2.5.13}{\cong} S\langle s/\pi^q \rangle$ for the second half. □

2.6 Categories of sheaves

Fix a Banach ring R and a site X.

We assume that the reader is comfortable with Schneiders' formalism of sheaves valued in quasi-abelian categories, cf. [52]. See also Appendix B.

Lemma 2.6.1. *Suppose all coverings in X are finite. A $\mathbf{Ban}_R$-presheaf on X is a sheaf if and only if the following sequence is strictly exact for every open $U \in X$ and every covering $\mathfrak{U}$ of U:*

$$0 \to \mathcal{F}(U) \to \prod_{V \in \mathfrak{U}} \mathcal{F}(V) \to \prod_{W, W' \in \mathfrak{U}} \mathcal{F}(W \times_U W').$$

Proof. The implication $\Rightarrow$ is clear. $\Leftarrow$ follows because $\mathbf{Ban}_R$ has finite products. □

Lemma 2.6.2. *Given a sheaf $\mathcal{F}\colon X \to \mathbf{Ban}_R$, its composition with the canonical functor $\mathbf{Ban}_R \to \mathbf{IndBan}_R$ is a sheaf of R-ind-Banach modules.*

Proof. Lemma 2.2.4 applies because $\mathbf{Ban}_R$ has finite limits. □

Recall the definition of the functor (2.2.1).

Definition 2.6.3. Given a sheaf $\mathcal{F}$ of R-ind-Banach algebras on X, $|\mathcal{F}|$ is the sheafification of the presheaf $U \mapsto |\mathcal{F}(U)|$ of abstract $|R|$-modules.

Lemma 2.6.4. *Given a sheaf $\mathcal{F}$ of R-ind-Banach algebras on X, the canonical map $|\mathcal{F}(U)| \xrightarrow{\cong} |\mathcal{F}|(U)$ is an isomorphism of abstract $|R|$-modules if X has only finite coverings.*

Proof. This follows from Lemma 2.2.7. □

Notation 2.6.5. $\mathcal{R}$ is a monoid in the category of sheaves on X with values in a closed symmetric monoidal quasi-abelian category $\mathbf{E}$. $\mathcal{M}$ is an $\mathcal{R}$-module object.

(i) $\mathcal{R}$ is a *sheaf of S-Banach algebras* if $\mathbf{E} = \mathbf{Ban}_S$, where S is an R-Banach algebra. $\mathcal{M}$ is a *sheaf of $\mathcal{R}$-Banach modules.*

(ii) $\mathcal{R}$ is a *sheaf of S-ind-Banach algebras* if $\mathbf{E} = \mathbf{IndBan}_S$, where S is an R-ind-Banach algebra. $\mathcal{M}$ is a *sheaf of $\mathcal{R}$-ind-Banach modules.*

Chapter 3

Period sheaves

We introduce the overconvergent de Rham period sheaf and the overconvergent de Rham period structure sheaf. Our constructions are inspired by [54, section 6].

3.1 Conventions and notation

Throughout, fix a prime number p and a perfect field κ of characteristic p. Write $k_0 := W(\kappa)[1/p]$, and let k denote a finite extension of k_0. The absolute value on k_0 extends to an absolute value on k cf. [21, Appendix A, Theorem 3]. Because k_0 is discretely valued, k is discretely valued. Fix a uniformiser $\pi \in k$. $k^\circ \subseteq k$ is the ring of power-bounded elements. C is the completion of a fixed algebraic closure of k.

This set-up allows any finite extension k of $\mathbb{Q}_p$, and in this case κ is a finite field. If κ is an algebraic closure of $\mathbb{F}_p$, then we may choose $k = k_0$, the maximal unramified extension of $\mathbb{Q}_p$.

3.2 The pro-étale site

We recall the pro-étale site associated to a smooth locally Noetherian adic space X over $\mathrm{Spa}(k, k^\circ)$ from [54]. Regarding the theory of adic spaces, we follow the notation given in [57, Lecture 2 and 3]. See the [54, beginning of section 3] for the definition of *locally Noetherian*.

$\mathbf{Pro}\,(X_{\text{ét}})$ is the pro-completion of the category $X_{\text{ét}}$ of adic spaces which are étale over X. The underlying topological space of "$\varprojlim$"$_{i\in I} U_i \in \mathbf{Pro}\,(X_{\text{ét}})$ is $\varprojlim_{i\in I} |U_i|$, where $|U_i|$ is the underlying topological space of U_i. $U \in \mathbf{Pro}\,(X_{\text{ét}})$ is *pro-étale over* X if and only if it is isomorphic to an object "$\varprojlim$"$_{i\in I} U_i$ such that all transition maps $U_j \to U_i$ are finite étale and surjective.

The *pro-étale site* $X_{proét}$ *of* X is the full subcategory of $\mathbf{Pro}(X_{\text{ét}})$ consisting of objects which are pro-étale over X. A collection of maps $\{f_i\colon U_i \to U\}$ in $X_{\text{proét}}$ is a covering if and only if the collection $\{|U_i| \to |U|\}$ is a pointwise covering of the topological space $|U|$, and a second set-theoretic condition is satisfied, see [55].

$\nu\colon X_{\text{proét}} \to X_{\text{ét}}$ is the canonical projection of sites.

(i) Let $U \in X_{\text{proét}}$. By [54, Lemma 4.2(iii)], $\varprojlim_{j\geq 1} \nu^{-1}\mathcal{O}^+_{X_{\text{ét}}}(U)/p^j$ is p-adically complete, thus it is π-adically complete. Equip it with the π-adic norm, cf. Definition 2.1.2. This makes $\varprojlim_{j\geq 1} \nu^{-1}\mathcal{O}^+_{X_{\text{ét}}}(U)/p^j$ a k°-Banach algebra. It is thus a k°-ind-Banach algebra. The sheafification of

$$U \mapsto \varprojlim_{j\geq 1} \nu^{-1}\mathcal{O}^+_{X_{\text{ét}}}(U)/p^j$$

is the *completed integral structure sheaf* $\widehat{\mathcal{O}}^+_{X_{\text{proét}}}$. It is a sheaf of k°-ind-Banach algebras because sheafification is strongly monoidal, cf. Lemma B.1.5.

(ii) Consider the presheaf of k-Banach algebras

$$U \mapsto \left(\varprojlim_{j\geq 1} \nu^{-1}\mathcal{O}^+_{X_{\text{ét}}}(U)/p^j\right) \widehat{\otimes}_{k^\circ} k \overset{2.3.6}{\cong} \left(\varprojlim_{j\geq 1} \nu^{-1}\mathcal{O}^+_{X_{\text{ét}}}(U)/p^j\right)[1/\pi].$$

Sheafify it as a presheaf of k-ind-Banach algebras to get the *completed structure sheaf* $\widehat{\mathcal{O}}_{X_{\text{proét}}}$. It is a sheaf of k-ind-Banach algebras by Lemma B.1.5.

Notation 3.2.1. We may write $\widehat{\mathcal{O}}^+ = \widehat{\mathcal{O}}^+_{X_{\text{proét}}}$ and $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{X_{\text{proét}}}$.

Recall [54, Definition 4.3]. K is a perfectoid field of characteristic zero, cf. [53, Definition 3.1], containing a ring of integral elements $K^+ \subseteq K$, cf. [57, Definition 2.2.12]. The definition of the pro-étale site still makes sense for a given locally Noetherian adic space Y over $\operatorname{Spa}(K, K^+)$. $U \in Y_{\text{proét}}$ is *affinoid perfectoid* if it is isomorphic to "$\varprojlim$"$_{i\in I} U_i$ with $U_i = \operatorname{Spa}(R_i, R_i^+)$ such that, denoting by R^+ the p-adic completion of $\varinjlim_{i\in I} R_i^+$ and $R = R^+[1/p]$, the pair (R, R^+) is a perfectoid affinoid (K, K^+)-algebra, cf. [53, Definition 5.1(i)].

Notation 3.2.2. With the notation from the previous paragraph, $\widehat{U} := \operatorname{Spa}(R, R^+)$.

Consider again X, the smooth locally Noetherian adic space over $\operatorname{Spa}(k, k^\circ)$. $U \in X_{\text{proét}}$ is *affinoid perfectoid* if the structural map $U \to X$ factors through a pro-étale map $X_K \to X$ such that $U \in X_{\text{proét}}/X_K \simeq X_{K,\text{proét}}$ is affinoid perfectoid; see [54, Proposition 3.15] for the canonical identification of the sites. Here, K denotes the completion of an algebraic extension of k which is perfectoid. We further fix a ring of integral elements $K^+ \subseteq K$ containing k°. We consider the pair (K, K^+) as an object $V = \text{“}\varprojlim\text{”}_{i \in I} V_i \in \operatorname{Spa}(k, k^\circ)_{\text{proét}}$ with $V_i = \operatorname{Spa}(K_i, K_i^+)$ such that K^+ is the π-adic completion of $\varinjlim_{i \in I} K_i^+$. The base change $X_K := X_{(K,K^+)} := X \times_{\operatorname{Spa}(k,k^\circ)} V$ is then an object in $X_{\text{proét}}$.

The following Lemma 3.2.3 appears implicitly in [54].

Lemma 3.2.3. *The affinoid perfectoids form a basis for the site $X_{proét}$.*

Proof. Every finite field extension $k \subseteq k' \subseteq C$ induces a finite étale map $\operatorname{Sp} k' \to \operatorname{Sp} k$ between the associated rigid-analytic spaces. Thus it induces a finite étale map between the associated adic spaces $\operatorname{Spa}(k', k'^\circ) \to \operatorname{Sp}(k, k^\circ)$, cf. [37, Proposition 1.7.11]. In particular, $V = \text{“}\varprojlim\text{”}_{[k':k]<\infty} (k', k'^\circ) \in \operatorname{Spa}(k, k^\circ)_{\text{proét}}$. Set $C^+ := \widehat{\varinjlim_{[k':k]<\infty} k'^\circ} \subseteq C$. C is perfectoid. By the [36, remark proceeding Proposition 3.9], $C^+ \subseteq C$ is a ring of integral elements. By construction, $X_C := X_{(C,C^+)} \to X$ is a covering. Thus every covering of X_C by affinoid perfectoids will give a covering of X by affinoid perfectoids. Now apply [54, Corollary 4.7]. $\square$

Consider the full subcategory $X_{\text{proét,affperfd}} \subseteq X_{\text{proét}}$ of affinoid perfectoids.

Lemma 3.2.4. *$X_{proét,affperfd}$ is closed under fibre products.*

Proof. Fix a diagram $U_1 \to U_2 \leftarrow U_3$ of affinoid perfectoids in $X_{\text{proét}}$ that lives over a fixed perfectoid affinoid field (K, K^+) of characteristic zero. Recall Notation 3.2.2. The fibre product $\widehat{U_1} \times_{\widehat{U_2}} \widehat{U_3}$ exists in the category of adic spaces and is again a perfectoid space, see [53, Proposition 6.18]. On the other hand, $U_1 \times_{U_2} U_3$ is perfectoid by [54, Lemma 4.6], and the universal property of the fibre product yields a map

$$\widehat{U_1 \times_{U_2} U_3} \to \widehat{U_1} \times_{\widehat{U_2}} \widehat{U_3}. \tag{3.2.1}$$

We claim that it is an isomorphism of adic spaces.

Write $\widehat{U_1} = \operatorname{Spa}\left(R_1, R_1^+\right)$, $\widehat{U_2} = \operatorname{Spa}\left(R_2, R_2^+\right)$, $\widehat{U_3} = \operatorname{Spa}\left(R_3, R_3^+\right)$, and $\widehat{U_1} \times_{\widehat{U_2}} \widehat{U_3} = \operatorname{Spa}(S, S^+)$. Here $S = R_1 \widehat{\otimes}_{R_2} R_3$ and S^+ is the completion of the integral closure of the image of $R_1^+ \otimes_{R_2^+} R_3^+$ in S. On the other hand, write the diagram $U_1 \to U_2 \leftarrow U_3$ above, after suitable reindexing as explained in [44, page 54, remark 1.133], as an inverse limit $U_{1i} \to U_{2i} \leftarrow U_{3i}$ of affinoids over a small cofiltered category I. Fix notation $U_{1i} = \operatorname{Spa}\left(R_{1i}, R_{1i}^+\right)$, $U_{2i} = \operatorname{Spa}\left(R_{2i}, R_{2i}^+\right)$, and $U_{3i} = \operatorname{Spa}\left(R_{3i}, R_{3i}^+\right)$. Assume that all these Huber pairs are complete. Then each $U_{1i} \times_{U_{2i}} U_{3i}$ exists in the category of adic spaces, see [37, Proposition 1.2.2], and $U_1 \times_{U_2} U_3 = \text{“}\varprojlim\text{”}_i U_{1i} \times_{U_{2i}} U_{3i}$. Indeed, *loc. cit.* says that $U_{1i} \times_{U_{2i}} U_{3i} = \operatorname{Spa}(S_i, S_i^+)$ where $S_i = R_{1i} \otimes_{R_{2i}} R_{3i}$, S_i^+ is the integral closure of $R_{1i}^+ \otimes_{R_{2i}^+} R_{3i}^+$ in S_i, and both S_i and S_i^+ carry suitabale topologies. Now write $S_\infty := \varinjlim_i S_i$ and $S_\infty^+ := \varinjlim_i S_i^+$. We turn S_∞ into a Huber ring by declaring S_∞^+ to be a ring of definition, which we equip with the π-adic topology. This gives a Huber pair (S_∞, S_∞^+). To check this, we have to verify three conditions:

(i) $S_\infty^+ \subseteq S_\infty$ is by definition open.

(ii) $S_\infty^+ \subseteq S_\infty^\circ$ follows from [57, Proposition 2.2.10.2].

(iii) It remains to show that S_∞^+ is integrally closed in S_∞. To show this, consider an element $s \in S_\infty$ such that there exists a monic polynomial $f = \sum_{j=0}^n f_i L^i \in S_\infty^+[L]$ with $f(s) = 0$. Pick indices l and $i_0, \ldots, i_n$ such that $s \in S_l$ and $f_j \in S_{i_j}^+$ for all $j = 0, \ldots, n$. Recall that the index category I was assumed to be cofiltered. That is, I^{op} is filtered and thus we find an element $t \in I^{\mathrm{op}}$ together with morphisms from l and $i_0, \ldots, i_n$ to t. In particular, $s \in S_t$ and $f \in S_t^+[L]$. Because $S_t^+ \subseteq S_t$ is integrally closed, $s \in S_t^+$ follows, and therefore $s \in S_\infty$.

We observe that the map (3.2.1) above is given by $\mathrm{Spa}\left(\widehat{(S_\infty, S_\infty^+)}\right) \to \mathrm{Spa}(S, S^+)$. It is an isomorphism by construction. □

Definition 3.2.5. Fix $U \in X_{\text{proét}}$.

(i) $X_{\text{proét,affperfd}}/U$ is the full subcategory of $X_{\text{proét}}/U$ whose objects are the maps $V \to U$ for affinoid perfectoid V. We equip it with the induced topology, and Lemma 3.2.4 shows that it gives rise to a site.

(ii) $X^{\text{fin}}_{\text{proét,affperfd}}/U$ is the site whose underlying category is the category underlying $X_{\text{proét,affperfd}}/U$, but we consider only the finite coverings.

If $U = X$, write $X_{\text{proét,affperfd}} := X_{\text{proét,affperfd}}/X$ and $X^{\text{fin}}_{\text{proét,affperfd}} := X^{\text{fin}}_{\text{proét,affperfd}}/X$.

Lemma 3.2.6. *Fix a covering $U \to X$ in $X_{\text{proét}}$. Then the morphisms of sites*

$$\begin{array}{ccccc} X_{\textit{proét}} & \longrightarrow & X_{\textit{proét,affperfd}} & \longrightarrow & X^{\textit{fin}}_{\textit{proét,affperfd}} \\ \uparrow & & \uparrow & & \uparrow \\ X_{\textit{proét}}/U & \longrightarrow & X_{\textit{proét,affperfd}}/U & \longrightarrow & X^{\textit{fin}}_{\textit{proét,affperfd}}/U \end{array}$$

give rise to equivalences of categories

$$\begin{array}{ccccc} \mathrm{Sh}\left(X_{\textit{proét}}, \mathbf{E}\right) & \xrightarrow{\simeq} & \mathrm{Sh}\left(X_{\textit{proét,affperfd}}, \mathbf{E}\right) & \xrightarrow{\simeq} & \mathrm{Sh}\left(X^{\textit{fin}}_{\textit{proét,affperfd}}, \mathbf{E}\right) \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Sh}\left(X_{\textit{proét}}/U, \mathbf{E}\right) & \xrightarrow{\simeq} & \mathrm{Sh}\left(X_{\textit{proét,affperfd}}/U, \mathbf{E}\right) & \xrightarrow{\simeq} & \mathrm{Sh}\left(X^{\textit{fin}}_{\textit{proét,affperfd}}/U, \mathbf{E}\right) \end{array}$$

for any elementary quasi-abelian category $\mathbf{E}$.

Proof. The vertical morphism at the left-hand side is an equivalence because $U \to X$ is a covering. It remains to check that the horizontal arrows are equivalences. Since the row at the top is obtained from the row at the bottom by setting $U = X$, it suffices to check that the horizontal morphisms at the bottom are equivalences. Lemma 3.2.3 gives the first equivalence $\mathrm{Sh}\,(X_{\text{proét}}/U, \mathbf{E}) \simeq \mathrm{Sh}\,(X_{\text{proét,affperfd}}/U, \mathbf{E})$. The second equivalence $\mathrm{Sh}\,(X_{\text{proét,affperfd}}/U, \mathbf{E}) \simeq \mathrm{Sh}\left(X^{\text{fin}}_{\text{proét,affperfd}}/U, \mathbf{E}\right)$ follows from Lemma B.2.1, which applies because all affinoid perfectoids are quasicompact objects in $X_{\text{proét}}$, cf. [54, Proposition 3.12], □

Finally, we give a local description of the structure sheaves.

Lemma 3.2.7. *Assume that* $U \in X_{\text{proét}}$ *is affinoid perfectoid with* $\widehat{U} = \mathrm{Spa}(R, R^+)$. *Equip* R^+ *with the* p*-adic norm, giving* $R = R^+[1/\pi]$ *the structure of* k*-Banach algebra. Then* $\widehat{\mathcal{O}}^+(U) \cong R^+$ *and* $\widehat{\mathcal{O}}(U) \cong R$.

Proof. Thanks to Lemma 3.2.6, we may view $\widehat{\mathcal{O}}^+$ and $\widehat{\mathcal{O}}$ as sheaves on the site $X^{\text{fin}}_{\text{proét,affperfd}}$. By [54, Lemma 4.10(iii)], it suffices to show that the presheaves $U \mapsto R^+$ and $U \mapsto R$ are sheaves.

Loc. cit. says that $U \mapsto R$ is a sheaf of abstract k-algebras. The open mapping theorem implies that it is a sheaf of k-Banach algebras, and it is a sheaf of k-ind-Banach algebras by Lemma 2.6.2.

We know by [54, Lemma 4.10(iii)] that $U \mapsto R^+$ is a sheaf of abstract k°-algebras. $U \mapsto R$ being a sheaf of k-Banach algebras implies that $U \mapsto R^+$ is a sheaf of k°-Banach algebras. Another application of Lemma 2.6.2 finishes the proof. □

3.3 The overconvergent de Rham period ring

This subsection follows the discussion at the beginning of [54, section 6]. Our new contribution is the definition of the overconvergent de Rham period ring. Fix the completion K of an algebraic extension of k which is perfectoid. Pick a ring of integral elements $K^+ \subseteq K$ containing k° and recall *tilting*, cf. [53, Lemma 3.4]. Fix an element $p^\flat \in K^\flat$ such that $\left(p^\flat\right)^\sharp/p \in (K^+)^\times$. Let (R, R^+) denote an affinoid perfectoid (K, K^+)-algebra. Its tilt is $\left(R^\flat, R^{\flat+}\right)$, cf. [53, Proposition 5.17 and Lemma 6.2].

3.3.1 Integral period rings

$$\mathbb{A}_{\inf}\left(R, R^{+}\right) := W\left(R^{\flat +}\right),$$

is the *relative infinitesimal period ring*. Here the operator W refers to the (always p-typical) Witt vectors. We equip $\mathbb{A}_{\inf}(R, R^{+})$ with the $\left(p, \left[p^{\flat}\right]\right)$-adic seminorm, cf. Definition 2.1.2. That is, $\|x\| := p^{-v}$, where $v \in \mathbb{N}$ is maximal with respect to the property that $x \in \left(p, \left[p^{\flat}\right]\right)^{v}$; $\|x\| := 0$ if $x \in \left(p, \left[p^{\flat}\right]\right)^{v}$ for all $v \in \mathbb{N}$. When the underlying perfectoid affinoid field is understood, write $A_{\inf} := \mathbb{A}_{\inf}(K, K^{+})$.

Lemma 3.3.1. *The underlying abstract ring of* $\mathbb{A}_{\inf}(R, R^{+})$ *is a strict* p*-ring.*

Proof. $R^{\flat}$ is a perfectoid $K^{\flat}$-algebra, see the discussion in [53, section 5]. *Loc. cit.* Proposition 5.9 gives that $R^{\flat +}$ is a perfect $\mathbb{F}_p$-algebra and thus the lemma. □

Since κ is perfect, we get the isomorphism at the left of the composition

$$\kappa \cong \varprojlim_{\Phi} \kappa = \varprojlim_{\Phi} k^{\circ}/\pi \to \varprojlim_{\Phi} K^{+}/\pi \cong K^{\flat +}. \tag{3.3.1}$$

Here, Φ denotes the Frobenii. The isomorphism at the right-hand side comes from [53, Lemma 3.4]. This composition (3.3.1) is a morphism of rings, thus it gives a map

$$W(\kappa) \to A_{\inf}$$

between the associated rings of Witt vectors. We find that $A_{\inf}$ is a $W(\kappa)$-algebra, and so is $\mathbb{A}_{\inf}(R, R^{+})$. Once we equip $W(\kappa)$ with the p-adic norm, Lemma 2.1.3 implies that they are both $W(\kappa)$-Banach algebras.

On the other hand, k° is a $W(\kappa)$-algebra and so is K^{+}. Lemma 2.1.3 implies that K^{+} is a $W(\kappa)$-Banach algebra, and so is R^{+}. We may now follow the [26, proof of Proposition 4.4.2] to find that *Fontaine's map*

$$\begin{aligned} \theta_{\inf} \colon \mathbb{A}_{\inf}\left(R, R^{+}\right) &\to R^{+}, \\ \sum_{n \geq 0} [a_n] p^{n} &\mapsto \sum_{n \geq 0} a_n^{\sharp} p^{n}. \end{aligned}$$

is a morphism of $W(\kappa)$-algebras. We will see shortly, cf. Lemma 3.3.4, that $\theta_{\inf}$ is a morphism of $W(\kappa)$-Banach algebras.

Lemma 3.3.2. *There is an element* $\xi \in A_{\inf}$ *that generates* $\ker \theta_{\inf}$ *and is not a zero-divisor in* $\mathbb{A}_{\inf}(R, R^{+})$*. It is of the form* $\xi = \left[p^{\flat}\right] - ap$ *for some unit* $a \in A_{\inf}$*.*

Proof. Everything is proven in [54, Lemma 6.3], except that a is a unit. Write $\xi = (\xi_0, \xi_1, \xi_2, \dots)$ as a Witt vector. Then ξ_1 is a unit in $K^{\flat+}$, see [16, Remark 3.11]. That is $\xi = \sum_{n\geq 0} \left[\xi_n^{1/p^n}\right] p^n$, thus $a = -\sum_{n\geq 0} \left[\xi_{n+1}^{1/p^{n+1}}\right] p^n$ is a unit modulo p. Because A_{inf} is p-adically complete, by Lemma 3.3.1, the result follows with [59, Tag 05GI]. □

We remark that the definition of ξ requires a choice, see [16, Remark 3.11].

Corollary 3.3.3. *$\mathbb{A}_{\text{inf}}(R, R^+)$ is a $W(\kappa)$-Banach algebra.*

We learned the following proof from [43].

Proof of Corollary 3.3.3. We have to check that $\mathbb{A}_{\text{inf}}(R, R^+)$ is complete with respect to the $(p, [p^\flat])$-adic topology. But $(p, [p^\flat]) = (p, \xi)$, and ξ, p is a regular sequence: ξ is not a zero-divisor, by Lemma 3.3.2, and p is not a zero-divisor in $\mathbb{A}_{\text{inf}}(R, R^+)/\xi \cong R^+$. Thus Lemma C.0.2 applies so that it suffices to check that $\mathbb{A}_{\text{inf}}(R, R^+)$ is ξ-adically and p-adically complete. Now see the [34, proof of Proposition 15.3.4]. □

Corollary 3.3.4. *θ_{inf} is a strict epimorphism of $W(\kappa)$-Banach algebras.*

Proof. Surjectivity is clear. Lemma 3.3.2 implies $(p, [p^\flat]) = (p, \xi) = (p) + \ker \theta_{\text{inf}}$. Everything follows now from the Lemmata 2.1.3 and 2.1.7. □

Lemma 3.3.5. *Fix an element $f \in \mathbb{A}_{\text{inf}}(R, R^+)$. (i) If ξ divides fp then ξ divides f. (ii) If p divides $f\xi$ then p divides f.*

Proof. Let ξ divide fp. We get $0 = \theta_{\text{inf}}(fp) = \theta_{\text{inf}}(f)p \in R^+$. That is $\theta_{\text{inf}}(f) = 0$, giving $\xi | f$ and thus (i). To prove (ii), assume that p divides $f\xi$. Consider $\sigma\colon \mathbb{A}_{\text{inf}}(R, R^+) \to \mathbb{A}_{\text{inf}}(R, R^+)/p \cong R^{\flat+}$. Then $0 = \sigma(f\xi) = \sigma(f[p^\flat]) = \sigma(f)p^\flat \in R^{\flat,+}$. Thus $\sigma(f) = 0$, giving $p|f$. □

Lemma 3.3.6. *For every $f \in \mathbb{A}_{\text{inf}}(R, R^+)$, (i) $\|f\xi\| = \|f\|p^{-1}$, (ii) $\|fp\| = \|f\|p^{-1}$.*

Proof. First, we prove (i). The estimate $\|f\xi\| \leq \|f\|p^{-1}$ is clear. Recall the description $(p, [p^\flat]) = (p, \xi)$ from Lemma 3.3.2. To show $\geq$, assume that $f \in (p, \xi)^v$, and v is maximal with respect to this property. Then we have to show that v is maximal with respect to the property $f\xi \in (p, \xi)^{v+1}$. Suppose it is not, that is $f\xi \in (p, \xi)^{v+2}$. We may write $f\xi = \sum_{i=0}^{v+2} f_i p^i \xi^{v+2-i}$ for certain $f_0, \dots, f_{v+2} \in \mathbb{A}_{\text{inf}}(R, R^+)$. Then $f_{v+2}p^{v+2}$ is divisible by ξ. Write $f_{v+2}p^{v+2} = f'_{v+2}p^{v+2}\xi$ for some f'_{v+2} with Lemma 3.3.5(i) such that $f\xi = \sum_{i=0}^{v+1} f_i p^i \xi^{v+2-i} + f'_{v+2}p^{v+2}\xi$. Since ξ is not a zero-divisor, we find the following contradiction:

$$f = \sum_{i=0}^{v+1} f_i p^i \xi^{v+1-i} + f'_{v+2}p^{v+2} \in (p, \xi)^{v+1}.$$

The proof of (ii) is similar, but we apply Lemma 3.3.5(ii) and use that p is not a zero-divisor, cf. Lemma 3.3.1. $\square$

Recall Definition 2.5.1 and Notation 2.4.1.

Definition 3.3.7. For all $q \in \mathbb{N}$, define the $W(\kappa)$-Banach algebra

$$\mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right) := \mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)\left\langle \frac{\ker \theta_{\mathrm{inf}}}{p^{q}} \right\rangle.$$

For $q = \infty$, define the $W(\kappa)$-ind-Banach algebra

$$\mathbb{A}_{\mathrm{dR}}^{\infty}\left(R, R^{+}\right) := \mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)\left\langle \frac{\ker \theta_{\mathrm{inf}}}{p^{\infty}} \right\rangle = \text{“}\varinjlim_{q\in\mathbb{N}}\text{”}\mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right).$$

Notation 3.3.8. Write $\mathbb{A}_{\mathrm{dR}}^{\dagger}\left(R, R^{+}\right) := \mathbb{A}_{\mathrm{dR}}^{\infty}\left(R, R^{+}\right) = \text{“}\varinjlim\text{”}_{q\in\mathbb{N}}\mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right)$.

We highlight that $\mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)$ carries the $\left(p, \left[p^{\flat}\right]\right)$-adic topology, which is equivalent to the (p, ξ)-adic topology by Lemma 3.3.2. For example, the power series $\sum_{\alpha\geq 0} \xi^{\alpha}\left(\zeta/p^{q}\right)^{\alpha}$ is an element of $\mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)\left\langle \zeta/p^{q}\right\rangle^{\alpha}$ with image

$$\sum_{\alpha\geq 0} \frac{\xi^{2\alpha}}{p^{q\alpha}} \in \mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right).$$

Lemma 3.3.9. *The canonical morphisms*

$$\mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)\left\langle \frac{\xi}{p^{q}} \right\rangle \xrightarrow{\cong} \mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right)$$

are isomorphisms of $\mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)$-Banach algebras for every $q \in \mathbb{N}$. It is an isomorphism of $\mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)$-ind-Banach algebras for $q = \infty$.

Proof. This follows from the Lemmata 2.5.10 and 3.3.2. $\square$

Lemma 3.3.10. *$\theta_{\mathrm{inf}}\colon \mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right) \to R^{+}$ factors through strict epimorphisms*

$$\theta_{\mathrm{dR}}^{q}\colon \mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right) \to R^{+}$$

of $W(\kappa)$-Banach algebras for every $q \in \mathbb{N}$. Their kernels are principal ideals generated by ξ/p^{q}. We exhibit a strict epimorphism

$$\theta_{\mathrm{dR}}^{\infty}\colon \mathbb{A}_{\mathrm{dR}}^{\infty}\left(R, R^{+}\right) \to R^{+}$$

of $W(\kappa)$-ind-Banach algebras by passing to the colimit along $q \to \infty$.

Abusing notation, we refer to θ_{dR}^{q} for $q \in \mathbb{N} \cup \{\infty\}$ again as *Fontaine's maps*.

Notation 3.3.11. Write $\theta_{\mathrm{dR}}^{\dagger} := \theta_{\mathrm{dR}}^{\infty}$.

Proof of Lemma 3.3.10. Fix q. Colimits preserve strict epimorphisms, cf. [14, Lemma 3.7], thus one may assume $q < \infty$. We get θ_{dR}^{q} from an application of the Lemmata 2.5.9 and 3.3.9. The proof of *loc. cit.* also gives a commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right) & & \\ {\scriptstyle\iota}\uparrow & \searrow^{\theta_{\mathrm{dR}}^{q}} & \\ \mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right) & \xrightarrow{\theta_{\mathrm{inf}}} & R^{+}, \end{array} \tag{3.3.2}$$

where ι is the canonical map. Since θ_{inf} is a strict epimorphism, cf. Lemma 3.3.4, [52, Proposition 1.1.8] implies that θ_{dR}^{q} is a strict epimorphism.

Now compute the kernel. Let $a := \sum_{\alpha \geq 0} a_{\alpha} \left(\xi / p^{q}\right)^{\alpha} \in \ker \theta_{\mathrm{dR}}^{q}$ with coefficients $a_{\alpha} \in \mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)$ for all $\alpha \geq 0$. Since θ_{dR}^{q} is bounded and $\theta_{\mathrm{inf}}(\xi) = 0$,

$$\theta_{\mathrm{dR}}^{q}\left(\sum_{\alpha \geq 1} a_{\alpha}\left(\frac{\xi}{p^{q}}\right)^{\alpha}\right) = 0. \tag{3.3.3}$$

Furthermore,

$$\theta_{\mathrm{inf}}\left(a_{0}\right) \overset{(3.3.5)}{=} \theta_{\mathrm{dR}}^{q}\left(\iota\left(a_{0}\right)\right) \overset{(3.3.3)}{=} \theta_{\mathrm{dR}}^{q}(a) = 0.$$

Lemma 3.3.2 implies $a_{0} = \widetilde{a}_{0} \xi$ for some $\widetilde{a}_{0} \in \mathbb{A}_{\mathrm{inf}}\left(R, R^{+}\right)$. We find

$$a = a_{0} + \sum_{\alpha \geq 1} a_{\alpha}\left(\frac{\xi}{p^{q}}\right)^{\alpha} = \frac{\xi}{p^{q}}\left(\widetilde{a}_{0} p^{q} + \sum_{\alpha \geq 1} a_{\alpha}\left(\frac{\xi}{p^{q}}\right)^{\alpha-1}\right) \in \left(\frac{\xi}{p^{q}}\right)$$

and $\ker \theta_{\mathrm{dR}}^{q} = \left(\xi / p^{q}\right)$ follows. □

Definition 3.3.12. Fix $q \in \mathbb{N}$. $\mathbb{A}_{\mathrm{dR}}^{>q}\left(R, R^{+}\right)$ is the completion of $\mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^{+}\right)$, equipped with the $(p, \ker \theta_{\mathrm{dR}}^{q})$-adic seminorm, cf. Definition 2.1.2.

Remark 3.3.13. We think of $\mathbb{A}_{\mathrm{dR}}^{>q}\left(R, R^{+}\right)$ as the functions given on an open tubular neighbourhood U around a vanishing locus $\{\xi = 0\}$, where ξ is a generator of $\ker \theta$. Intuitively, this neighbourhood has radius $|p|^{q}$, that is $U = \left\{x\colon \log_{p}\left(\operatorname{dist}(x, \xi)\right) > q\right\}$; this explain the superscript $> q$.

Lemma 3.3.14. *$\mathbb{A}_{\mathrm{dR}}^{>q}\left(R, R^{+}\right)$ is a $W(\kappa)$-Banach algebra for all $q \in \mathbb{N}$.*

Proof. One has to check that the $(p, \ker \theta_{\mathrm{dR}}^{q})$-adic completion is complete. This follows from [59, Tag 05GG], because $\ker \theta_{\mathrm{dR}}^{q}$ is a principal ideal by Lemma 3.3.10. □

Lemma 3.3.15. *The* $\mathbb{A}^q_{\mathrm{dR}}(R,R^+) \to \mathbb{A}^{q+1}_{\mathrm{dR}}(R,R^+)$ *factor canonically through the maps* $\mathbb{A}^q_{\mathrm{dR}}(R,R^+) \to \mathbb{A}^{>q}_{\mathrm{dR}}(R,R^+)$. *In particular, we get a canonical isomorphism*

$$\mathbb{A}^{\infty}_{\mathrm{dR}}\left(R,R^+\right) \xrightarrow{\cong} \text{“}\varinjlim_{q\in\mathbb{N}}\text{”}\mathbb{A}^{>q}_{\mathrm{dR}}\left(R,R^+\right)$$

of $W(\kappa)$*-ind-Banach algebras.*

Proof. We have to check that every element in the image of $(p, \ker\theta^q_{\mathrm{dR}})$ in $\mathbb{A}^{q+1}_{\mathrm{dR}}(R,R^+)$ is topologically nilpotent. By Lemma 3.3.10, it suffices to check this for p and ξ/p^q. This follows because p is already topologically nilpotent in $\mathbb{A}_{\mathrm{inf}}(R,R^+)$ and $\xi/p^q = p\cdot\xi/p^{q+1} \in \mathbb{A}^{q+1}_{\mathrm{dR}}(R,R^+)$. □

Remark 3.3.16. The rings $\mathbb{A}^{>q}_{\mathrm{dR}}(R,R^+)$ are ξ/p^q-adically complete, in contrast to the $\mathbb{A}^q_{\mathrm{dR}}(R,R^+)$. This becomes useful in the proof of Theorem 3.4.2.

Lemma 3.3.17. *The multiplication-by-*$(p^q\zeta-\xi)$*-map*

$$\mathbb{A}_{\mathrm{inf}}\left(R,R^+\right)\langle\zeta\rangle \to \mathbb{A}_{\mathrm{inf}}\left(R,R^+\right)\langle\zeta\rangle$$

is a strict monomorphism for every $q\in\mathbb{N}_{\geq 2}$. *Thus the* $(p^q\zeta-\xi) \subseteq \mathbb{A}_{\mathrm{inf}}(R,R^+)\langle\zeta\rangle$ *are closed and* $\mathbb{A}^q_{\mathrm{dR}}(R,R^+) \cong \mathbb{A}_{\mathrm{inf}}(R,R^+)\langle\zeta\rangle/(p^q\zeta-\xi)$.

Proof. We have to check that the morphism is injective, its image is closed, and it is open onto its image, cf. Lemma 2.1.7(iii).

To show injectivity, note that

$$(p^q\zeta-\xi)\sum_{\alpha\geq 0} a_\alpha\zeta^\alpha = 0$$

implies $-a_0\xi = 0$ and $-a_\alpha\xi + p^q a_{\alpha-1} = 0$ for all $\alpha\geq 1$. By induction and because ξ is not a zero-divisor, cf. Lemma 3.3.2, this gives $a_\alpha = 0$ for all α, thus injectivity.

Next, we show that the ideal $(p^q\zeta-\xi)\subseteq\mathbb{A}_{\mathrm{inf}}(R,R^+)\langle\zeta\rangle$ is closed. If $f=\sum_{n\geq 0} f_n$ is convergent in $\mathbb{A}_{\mathrm{inf}}(S,S^+)\langle\zeta\rangle$ with $f_n\in(p^q\zeta-\xi)$ for all $n\geq 0$, we can pick $g_n\in\mathbb{A}_{\mathrm{inf}}(S,S^+)\langle\zeta\rangle$ such that $f_n = g_n(p^q\zeta-\xi)$. Because $q\geq 2$, $p^{-q}<p^{-1}$. Thus

$$\|g_np^q\zeta\| \leq \|g_n\|p^{-q} < \|g_n\|p^{-1}.$$

On the other hand, Lemma 3.3.6 implies $\|g_n\xi\| = \|g_n\|p^{-1}$. This gives $\|g_np^q\zeta\| < \|g_n\xi\|$, therefore [21, section 2.1, Proposition 2] applies and we find

$$\|f_n\| = \|g_np^q\zeta - g_n\xi\| = \max\{\|g_np^q\zeta\|, \|g_n\xi\|\} = \|g_n\|p^{-1}.$$

Since $(f_n)_{n\geq 0}$ is a zero-sequence, this implies that $g_n \to 0$ for $n \to \infty$. In particular, $f = \left(\sum_{n\geq 0} g_n\right)(p^q\zeta - \xi) \in (p^q\zeta - \xi)$.

Regarding openness, this follows from the following fact, which we have already proven above: $\|g\,(p^q\zeta - \xi)\,\| = \|g\|p^{-q}$ for all $g \in \mathbb{A}_{\mathrm{inf}}\,(R, R^+)\,\langle\zeta\rangle$. Indeed, this implies that the ball of radius p^{-N} in the image of the multiplication-by-$(p^q\zeta - \xi)$-map is contained in the image of the pall of radius p^{-N+q}. Finally,

$$\mathbb{A}^q_{\mathrm{dR}}\left(R, R^+\right) \overset{3.3.9}{\cong} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right)\left\langle \frac{\xi}{p^q} \right\rangle \overset{2.5.13}{\cong} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right)\langle\zeta\rangle\,/\overline{(p^q\zeta - \xi)}$$

implies the second sentence of Lemma 3.3.17. □

Lemma 3.3.18. *$\xi/p^q \in \mathbb{A}^q_{\mathrm{dR}}\,(R, R^+)$ is not a zero-divisor for all $q \in \mathbb{N}_{\geq 2}$.*

Proof. The identification $\mathbb{A}^q_{\mathrm{dR}}\,(R, R^+) \cong \mathbb{A}_{\mathrm{inf}}\,(R, R^+)\,\langle\zeta\rangle\,/\,(p^q\zeta - \xi)$ from Lemma 3.3.17 implies that we have to check the following: Given $f = \sum_{\alpha\geq 0} f_\alpha \zeta^\alpha \in \mathbb{A}_{\mathrm{inf}}\,(R, R^+)\,\langle\zeta\rangle$,

$$\zeta f \in (p^q\zeta - \xi) \Rightarrow f \in (p^q\zeta - \xi)\,. \tag{3.3.4}$$

We compute that (3.3.4) holds. Let $g = \sum_{\alpha\geq 0} g_\alpha \zeta^\alpha$ such that

$$\eta f = g\,(p^q\eta - \xi) = -\xi g_0 + \sum_{\alpha\geq 1} (p^q g_{\alpha-1} - \xi g_\alpha)\,\zeta^\alpha.$$

Since ξ is not a zero-divisor, cf. Lemma 3.3.2, $g_0 = 0$. Proceeding by induction, we find $g_\alpha = 0$ for all $\alpha \geq 0$, thus $f = 0$. In particular, $f \in (p^q\zeta - \xi)$. □

Corollary 3.3.19. *Let $q \in \mathbb{N}_{\geq 2}$. The ring $\mathbb{A}^{>q}_{\mathrm{dR}}\,(R, R^+)$ is p-torsion free.*

Proof. We may check that the principal symbol $\sigma(p) \in \mathrm{gr}\,\mathbb{A}^{>q}_{\mathrm{dR}}\,(R, R^+)$ is not a zero-divisor, where we consider the associated graded with respect to the $(p, \xi/p^q)$-adic filtration, cf. [38, Chapter I, subsection 4.2 page 31-32, Theorem 4(5)]. To compute the associated graded, we first notice that it is canonically isomorphic to the associated graded of $\mathbb{A}^q_{\mathrm{dR}}\,(R, R^+)$, equipped with the $(p, \xi/p^q)$-adic filtration. The sequence $p, \xi/p^q \in \mathbb{A}^q_{\mathrm{dR}}\,(R, R^+)$ is regular: ξ/p^q is not a zero-divisor by Lemma 3.3.18, and the image of p in $\mathbb{A}^q_{\mathrm{dR}}\,(R, R^+)\,/\,(\xi/p^q) \cong R^+$ is not a zero-divisor as well, cf. Lemma 3.3.10. Now apply [28, Exercise 17.16.a] to get an isomorphism

$$\mathrm{gr}\,\mathbb{A}^{>q}_{\mathrm{dR}}\left(R, R^+\right) \cong \mathrm{gr}\,\mathbb{A}^q_{\mathrm{dR}}\left(R, R^+\right) \cong \left(R^+/p\right)\left[\sigma\,(p)\,, \sigma\left(\frac{\xi}{p^q}\right)\right],$$

where the principal symbols $\sigma\,(p)$ of p and $\sigma\,(\xi/p^q)$ of ξ/p^q are homogenous of degree 1. This proves Corollary 3.3.19. □

Here is a variant of Lemma 3.3.10:

Lemma 3.3.20. *$\theta_{\inf}\colon \mathbb{A}_{\inf}(R, R^+) \to R^+$ factors through strict epimorphisms*

$$\theta_{\mathrm{dR}}^{>q}\colon \mathbb{A}_{\mathrm{dR}}^{>q}\left(R, R^+\right) \to R^+$$

of $W(\kappa)$-Banach algebras for all $q \in \mathbb{N}$. If $q \geq 2$, then their kernels are principal ideals generated by the ξ/p^q, which are furthermore non-zero divisors.

Proof. The map $\theta_{\mathrm{dR}}^{>q}$ is the completion of bounded linear map

$$\theta_{\mathrm{dR}}^{q}\colon \mathbb{A}_{\mathrm{dR}}^{q}\left(R, R^+\right) \to R^+,$$

where $\mathbb{A}_{\mathrm{dR}}^{q}(R, R^+)$ carries the $(p, \xi/p^q)$-adic topology and R^+ is equipped with the p-adic topology. In particular, we gave a commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{dR}}^{>q}(R, R^+) & & \\ \uparrow{\scriptstyle\iota} & \searrow^{\theta_{\mathrm{dR}}^{>q}} & \\ \mathbb{A}_{\inf}(R, R^+) & \xrightarrow{\theta_{\inf}} & R^+, \end{array} \tag{3.3.5}$$

where ι is the canonical map. Since $\theta_{\inf}$ is a strict epimorphism, cf. Lemma 3.3.4, [52, Proposition 1.1.8] implies that $\theta_{\mathrm{dR}}^{>q}$ is a strict epimorphism.

To prove the second statement, consider the sequence

$$0 \longrightarrow \mathbb{A}_{\mathrm{dR}}^{>q}\left(R, R^+\right) \xrightarrow{\xi/p^q\cdot} \mathbb{A}_{\mathrm{dR}}^{>q}\left(R, R^+\right) \xrightarrow{\theta_{\mathrm{dR}}^{>q}} R^+ \longrightarrow 0$$

By the proof of Corollary 3.3.19, its associated graded is

$$0 \longrightarrow \left(R^+/p\right)\left[\sigma(p), \sigma\left(\frac{\xi}{p^q}\right)\right] \xrightarrow{\sigma(\xi/p^q)\cdot} \left(R^+/p\right)\left[\sigma(p), \sigma\left(\frac{\xi}{p^q}\right)\right] \xrightarrow{\operatorname{gr}\theta_{\mathrm{dR}}^{>q}} \left(R^+/p\right)[\sigma(p)] \longrightarrow 0$$

It is exact, thus [38, Chapter I, subsection 4.2 page 31-32, Theorem 4(5)] applies. ☐

Here is a variant of Lemma 3.3.5(i).

Lemma 3.3.21. *Fix an element $f \in \mathbb{A}_{\mathrm{dR}}^{>q}(R, R^+)$, $q \in \mathbb{N}_{\geq 2}$. If ξ/p^q divides fp then ξ/p^q divides f.*

Proof. We get $0 = \theta_{\mathrm{dR}}^{>q}(fp) = \theta_{\mathrm{dR}}^{>q}(f)p \in R^+$. This gives $\theta_{\mathrm{dR}}^{>q}(f) = 0$, and Lemma 3.3.20 implies that ξ/p^q divides f. ☐

3.3.2 Inverting p

Apply Lemma A.0.2 to get the seminormed k_0-algebra

$$\mathbb{B}_{\text{inf}}\left(R,R^+\right) := \mathbb{A}_{\text{inf}}\left(R,R^+\right) \otimes_{W(\kappa)} k_0$$

and the k_0-Banach algebra

$$\widehat{\mathbb{B}}_{\text{inf}}\left(R,R^+\right) := \mathbb{A}_{\text{inf}}\left(R,R^+\right) \widehat{\otimes}_{W(\kappa)} k_0.$$

Remark 3.3.22. $\mathbb{B}_{\text{inf}}(R,R^+)$ is not complete, as it does not contain $\sum_{n\geq 0}\xi^{n+1}/p^n$.

Similarly, we introduce the $\widehat{\mathbb{B}}_{\text{inf}}(R,R^+)$-Banach algebras

$$\mathbb{B}_{\text{dR}}^{q,+}\left(R,R^+\right) := \mathbb{A}_{\text{dR}}^{q}\left(R,R^+\right) \widehat{\otimes}_{W(\kappa)} k_0 \text{ and}$$
$$\mathbb{B}_{\text{dR}}^{>q,+}\left(R,R^+\right) := \mathbb{A}_{\text{dR}}^{>q}\left(R,R^+\right) \widehat{\otimes}_{W(\kappa)} k_0$$

for all $q\in\mathbb{N}$. For $q=\infty$, we have the $\widehat{\mathbb{B}}_{\text{inf}}(R,R^+)$-ind-Banach algebras

$$\mathbb{B}_{\text{dR}}^{\infty,+}\left(R,R^+\right) := \mathbb{A}_{\text{dR}}^{\infty}\left(R,R^+\right) \widehat{\otimes}_{W(\kappa)} k_0 = \mathbb{A}_{\text{dR}}^{\dagger}\left(R,R^+\right) \widehat{\otimes}_{W(\kappa)} k_0,$$

cf. Notation 3.3.8. Write $\mathbb{B}_{\text{dR}}^{\dagger,+}(R,R^+) := \mathbb{B}_{\text{dR}}^{\infty,+}(R,R^+)$ and note that

$$\mathbb{B}_{\text{dR}}^{\dagger,+}\left(R,R^+\right) = \text{“}\varinjlim_{q\in\mathbb{N}}\text{”}\, \mathbb{B}_{\text{dR}}^{q}\left(R,R^+\right).$$

Definition 3.3.23. The $\widehat{\mathbb{B}}_{\text{inf}}(R,R^+)$-ind-Banach algebra $\mathbb{B}_{\text{dR}}^{\dagger,+}(R,R^+)$ is the *relative positive overconvergent de Rham period ring.* Whenever $(R,R^+)=(K,K^+)$ is a well-understood perfectoid field, we refer to $B_{\text{dR}}^{\dagger,+} := \mathbb{B}_{\text{dR}}^{\dagger,+}(K,K^+)$ as the *positive overconvergent de Rham period ring.*

Lemma 3.3.24. *The morphisms*

$$\widehat{\mathbb{B}}_{\text{inf}}\left(R,R^+\right)\left\langle\frac{\xi}{p^q}\right\rangle \xrightarrow{\cong} \mathbb{B}_{\text{dR}}^{q,+}\left(R,R^+\right)$$

are isomorphisms of $\widehat{\mathbb{B}}_{\text{inf}}(R,R^+)$-Banach algebras for every $q\in\mathbb{N}$. They are a isomorphisms of $\widehat{\mathbb{B}}_{\text{inf}}(R,R^+)$-ind-Banach algebras for $q=\infty$.

Proof. We may assume $q<\infty$ without loss of generality. Compute

$$\begin{aligned}
&\widehat{\mathbb{B}}_{\text{inf}}\left(R,R^+\right)\left\langle\frac{\xi}{p^q}\right\rangle\\
&= \operatorname{coker}\left(\widehat{\mathbb{B}}_{\text{inf}}\left(R,R^+\right)\left\langle\frac{\zeta}{p^q}\right\rangle \xrightarrow{\xi - p^q\frac{\zeta}{p^q}} \widehat{\mathbb{B}}_{\text{inf}}\left(R,R^+\right)\left\langle\frac{\zeta}{p^q}\right\rangle\right)\\
&\cong \operatorname{coker}\left(\mathbb{A}_{\text{inf}}\left(R,R^+\right)\left\langle\frac{\zeta}{p^q}\right\rangle \xrightarrow{\xi - p^q\frac{\zeta}{p^q}} \mathbb{A}_{\text{inf}}\left(R,R^+\right)\left\langle\frac{\zeta}{p^q}\right\rangle\right)\widehat{\otimes}_{W(\kappa)} k_0\\
&\cong \mathbb{A}_{\text{inf}}\left(R,R^+\right)\left\langle\frac{\xi}{p^q}\right\rangle \widehat{\otimes}_{W(\kappa)} k_0\\
&\cong \mathbb{B}_{\text{dR}}^{q,+}\left(R,R^+\right),
\end{aligned}$$

where we have used Lemma 3.3.9 in the last step. □

Recall Fontaine's maps $\theta_{\inf}$, θ^q_{dR} for all $q \in \mathbb{N}$, and $\theta^{\dagger}_{\mathrm{dR}}$. They induce morphisms

$$\widehat{\vartheta}_{\inf} \colon \widehat{\mathbb{B}}_{\inf}\left(R, R^+\right) \xrightarrow{\theta_{\inf}\widehat{\otimes}_{W(\kappa)} \mathrm{id}_{k_0}} R^+\widehat{\otimes}_{W(\kappa)} k_0 \xrightarrow{\cong} R$$

$$\vartheta^{q,+}_{\mathrm{dR}} \colon \mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \xrightarrow{\theta^q_{\mathrm{dR}}\widehat{\otimes}_{W(\kappa)} \mathrm{id}_{k_0}} R^+\widehat{\otimes}_{W(\kappa)} k_0 \xrightarrow{\cong} R,$$

$$\vartheta^{\dagger,+}_{\mathrm{dR}} \colon \mathbb{B}^{+}_{\mathrm{dR}}\left(R, R^+\right) \xrightarrow{\theta^{\dagger}_{\mathrm{dR}}\widehat{\otimes}_{W(\kappa)} \mathrm{id}_{k_0}} R^+\widehat{\otimes}_{W(\kappa)} k_0 \xrightarrow{\cong} R$$

of k_0-Banach, respectively k_0-ind-Banach algebras. We refer to them again as *Fontaine's maps.*

Lemma 3.3.25. *The maps* $\widehat{\vartheta}_{\inf}$, $\vartheta^{q,+}_{\mathrm{dR}}$, *and* $\vartheta^{\dagger,+}_{\mathrm{dR}}$, *are strict epimorphisms.*

Proof. We have to check that the maps $\theta_{\inf}\widehat{\otimes}_{W(\kappa)} \mathrm{id}_{k_0}$, $\theta^{q,+}_{\mathrm{dR}}\widehat{\otimes}_{W(\kappa)} \mathrm{id}_{k_0}$, and $\theta^{\dagger,+}_{\mathrm{dR}}\widehat{\otimes}_{W(\kappa)} \mathrm{id}_{k_0}$ are strict epimorphisms. But the completed tensor product preserves strict epimorphisms, cf. [14, Lemma 3.7]. Now apply Corollary 3.3.4 and Lemma 3.3.10. □

3.3.3 Inverting t, Fontaine's $2\pi i$

Assume that K admits a compatible system $1, \zeta_p, \zeta_{p^2}, \dots$ of primitive pth power roots of unity, that is $\zeta^p_{p^{n+1}} = \zeta_{p^n}$ for all $n \in \mathbb{N}$. We fix such a system. Since each ζ_{p^n} satisfies the monic polynomial $L^{p^n} - 1 \in L^+[X]$ and K^+ is integrally closed, it follows that this system $\{\zeta_{p^n}\}_{n\in\mathbb{N}}$ lies in K^+. Thus we have $\epsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in K^{\flat+}$. Furthermore,

$$\theta(\epsilon - 1) = 1 - 1 = 0. \tag{3.3.6}$$

Write $A^1_{\mathrm{dR}} := \mathbb{A}^1_{\mathrm{dR}}\left(K, K^+\right)$.

Definition 3.3.26. The computation (3.3.6) allows us to define

$$\begin{aligned} t := \log([\epsilon]) = \log(1 + ([\epsilon] - 1)) &= \sum_{\alpha \geq 1} (-1)^{\alpha+1} \frac{([\epsilon]-1)^\alpha}{\alpha} \\ &= \sum_{\alpha \geq 1} (-1)^{\alpha+1} \frac{p^\alpha}{\alpha} \left(\frac{[\epsilon]-1}{p}\right)^\alpha \in A^1_{\mathrm{dR}}. \end{aligned}$$

Here, we are using that α divides p^α in n in $\mathbb{Z}_p$.

Notation 3.3.27. Introduce the following $W(\kappa)$-ind-Banach algebra

$$\text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}} := \text{“}\varinjlim\text{”} \left(\cdots \xrightarrow{t\times} A^1_{\mathrm{dR}} \xrightarrow{t\times} A^1_{\mathrm{dR}} \xrightarrow{t\times} \dots\right).$$

The multiplication is

$$\text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}} \widehat{\otimes}_{W(\kappa)} \text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}} \cong \text{“}\varinjlim_{2t\times}\text{”} A^1_{\mathrm{dR}} \widehat{\otimes}_{W(\kappa)} A^1_{\mathrm{dR}} \xrightarrow{\text{“}\varinjlim\text{”}_{2t\times}\mu} \text{“}\varinjlim_{2t\times}\text{”} A^1_{\mathrm{dR}} \cong \text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}}$$

where μ is the multiplication on A^1_{dR}. The unit is induced by the unit $W(\kappa) \to A^1_{\mathrm{dR}}$.

Recall Notation 2.3.1 and define, for all $q \in \mathbb{N}$, the k_0-ind-Banach algebras

$$\begin{aligned}
\mathbb{B}^q_{\mathrm{dR}}\left(R, R^+\right) &:= \mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{A^1_{\mathrm{dR}}} \text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}} \text{ and} \\
\mathbb{B}^{>q}_{\mathrm{dR}}\left(R, R^+\right) &:= \mathbb{B}^{>q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{A^1_{\mathrm{dR}}} \text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}}.
\end{aligned}$$

For $q = \infty$, we define

$$\begin{aligned}
\mathbb{B}^\infty_{\mathrm{dR}}\left(R, R^+\right) &:= \mathbb{B}^{\infty,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{A^1_{\mathrm{dR}}} \text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}} \\
&= \mathbb{B}^{\dagger,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{A^1_{\mathrm{dR}}} \text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}}
\end{aligned}$$

Write $\mathbb{B}^\dagger_{\mathrm{dR}}(R, R^+) := \mathbb{B}^\infty_{\mathrm{dR}}(R, R^+)$.

Definition 3.3.28. $\mathbb{B}^\dagger_{\mathrm{dR}}(R, R^+)$ is the *relative overconvergent de Rham period ring.* When $(R, R^+) = (K, K^+)$ is a well-understood perfectoid field, we refer to $B^\dagger_{\mathrm{dR}} := \mathbb{B}^\dagger_{\mathrm{dR}}(K, K^+)$ as the *overconvergent de Rham period ring.*

The previous definitions depend a priori on the choice of ϵ.

Lemma 3.3.29. *Fix another choice $\epsilon' \in K^{\flat +}$ of a compatible system of primitive pth power roots of unity. Then there exists a unique unit $u \in \mathbb{A}_{\inf}$ such that $([\epsilon] - 1) = u([\epsilon'] - 1)$. Writing $t' := \log([\epsilon'])$, u induces an isomorphism*

$$\text{“}\varinjlim_{t\times}\text{”} A^1_{\mathrm{dR}} \cong \text{“}\varinjlim_{t'\times}\text{”} A^1_{\mathrm{dR}}. \tag{3.3.7}$$

As a consequence, the definitions of $\mathbb{B}^q_{\mathrm{dR}}(R, R^+)$ and $\mathbb{B}^{>q}_{\mathrm{dR}}(R, R^+)$ for all $q \in \mathbb{N}$ as well as $\mathbb{B}^\dagger_{\mathrm{dR}}(R, R^+)$ are independent of the choice of ϵ.

Proof. Write $\mu := [\epsilon] - 1$ and $\mu' := [\epsilon'] - 1$. The ideals $(\mu) = (\mu') \subseteq A_{\inf}$ coincide by [16, Lemma 3.23]. Proposition 3.17(ii) *loc. cit.* furthermore says that μ and μ' are non-zero-divisors, thus there exists a unit u such that $\mu = u\mu'$. Using again that μ and μ' are non-zero-divisors, one deduces that u is unique with respect to that property. Next, note that $\alpha + 1$ divides p^α in $\mathbb{Z}_p$ and define

$$v := \sum_{\alpha \geq 0} (-1)^\alpha \frac{p^\alpha}{\alpha + 1} \left(\frac{\mu}{p}\right)^\alpha \in A^1_{\mathrm{dR}},$$

satisfying $t = v\mu$. Note that $v \equiv 1 \mod \xi/p$, which is a unit. Since A^1_{dR} is complete with respect to the ξ/p-adic topology, cf. Lemma 3.3.9 and Proposition 2.5.11, [59, Tag 05GI] implies that v is a unit. Similarly, write $t' = \mu' v'$ for some unit $v' \in A^1_{\mathrm{dR}}$. In particular, $t = (vuv')\, t'$. Then vuv' is again a unit, giving rise to the isomorphism (3.3.7). ☐

The previous definitions depend a priori on the choice of (K, K^+).

Lemma 3.3.30. *Suppose (R, R^+) is an affinoid perfectoid (K, K^+)-algebra, as well as an affinoid perfectoid (L, L^+). Then, for all $q \in \mathbb{N}$,*

$$\begin{aligned} &\mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)} \text{“}\varinjlim_{t_K\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(K, K^+\right) \\ &\cong \mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(L,L^+)} \text{“}\varinjlim_{t_L\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(L, L^+\right) \quad \textit{and} \\ &\mathbb{B}^{>q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)} \text{“}\varinjlim_{t_K\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(K, K^+\right) \\ &\cong \mathbb{B}^{>q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(L,L^+)} \text{“}\varinjlim_{t_L\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(L, L^+\right). \end{aligned}$$

This implies

$$\begin{aligned} &\mathbb{B}^{\dagger,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)} \text{“}\varinjlim_{t_K\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(K, K^+\right) \\ &\cong \mathbb{B}^{\dagger,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(L,L^+)} \text{“}\varinjlim_{t_L\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(L, L^+\right). \end{aligned}$$

Both $t_K \in \mathbb{A}^1_{\mathrm{dR}}(K, K^+)$ and $t_L \in \mathbb{A}^1_{\mathrm{dR}}(L, L^+)$ are given as t in Definition 3.3.26.

Proof. C denotes an algebraically closed field, complete with respect to a non-trivial, non-Archimedean valuation. Fix embeddings $K, L \hookrightarrow C$ and compute

$$\begin{aligned} &\mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)} \text{“}\varinjlim_{t_K\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(K, K^+\right) \\ &\cong \text{“}\varinjlim_{t_K\times}\text{”}\, \mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \\ &\cong \mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(C,C^+)} \text{“}\varinjlim_{t_K\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(C, C^+\right). \end{aligned}$$

This is in fact an isomorphism of k-ind-Banach algebras. We do this computation again but over the base L to get

$$\begin{aligned} &\mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(L,L^+)} \text{“}\varinjlim_{t_L\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(L, L^+\right) \\ &\cong \text{“}\varinjlim_{t_L\times}\text{”}\, \mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \\ &\cong \mathbb{B}^{q,+}_{\mathrm{dR}}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(C,C^+)} \text{“}\varinjlim_{t_L\times}\text{”} \mathbb{A}^1_{\mathrm{dR}}\left(C, C^+\right) \end{aligned}$$

Similar computations work for $\mathbb{B}_{\mathrm{dR}}^{>q,+}$. Now apply Lemma 3.3.29. Compute the colimit along $q \to \infty$ to get the result for $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$. $\square$

3.4 The overconvergent de Rham period sheaf

Fix a locally Noetherian adic space X over $\mathrm{Spa}(k, k^\circ)$. The constructions in the previous section 3.3 are functorial, in the following sense. For an affinoid perfectoid $U \in X_{\mathrm{pro\acute{e}t}}$ with $\widehat{U} = \mathrm{Spa}\,(R, R^+)$, Proposition 3.2.7 gives $(R, R^+) = \left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right)$. Thus on $X_{\mathrm{pro\acute{e}t,affperfd}}^{\mathrm{fin}}$ and for all $q \in \mathbb{N}$, we get the presheaves

$$\begin{aligned} \mathbb{A}_{\mathrm{inf}}^{\mathrm{psh}} &: U \mapsto \mathbb{A}_{\mathrm{inf}}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right), \\ \mathbb{A}_{\mathrm{dR}}^{q,\mathrm{psh}} &: U \mapsto \mathbb{A}_{\mathrm{dR}}^{q}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right), \text{ and} \\ \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}} &: U \mapsto \mathbb{A}_{\mathrm{dR}}^{>q}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right) \end{aligned}$$

of $W(\kappa)$-Banach algebras,

$$\mathbb{A}_{\mathrm{dR}}^{\infty,\mathrm{psh}} : U \mapsto \mathbb{A}_{\mathrm{dR}}^{\dagger}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right)$$

of $W(\kappa)$-ind-Banach algebras,

$$\begin{aligned} \widehat{\mathbb{B}}_{\mathrm{inf}}^{\mathrm{psh}} &: U \mapsto \widehat{\mathbb{B}}_{\mathrm{inf}}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right), \\ \mathbb{B}_{\mathrm{dR}}^{q,+,\mathrm{psh}} &: U \mapsto \mathbb{B}_{\mathrm{dR}}^{q,+}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right), \text{ and} \\ \mathbb{B}_{\mathrm{dR}}^{>q,+,\mathrm{psh}} &: U \mapsto \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right) \end{aligned}$$

of k_0-Banach algebras, and

$$\mathbb{B}_{\mathrm{dR}}^{\infty,+,\mathrm{psh}} : U \mapsto \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right)$$

of k_0-ind-Banach algebras. Write $\mathbb{A}_{\mathrm{dR}}^{\dagger,\mathrm{psh}} := \mathbb{A}_{\mathrm{dR}}^{\infty,\mathrm{psh}}$ and $\mathbb{B}_{\mathrm{dR}}^{\dagger,+,\mathrm{psh}} := \mathbb{B}_{\mathrm{dR}}^{\infty,+,\mathrm{psh}}$.

Since k° does not contain a compatible system of pth power roots of unity, the element t does not exist on the whole site $X_{\mathrm{pro\acute{e}t,affperfd}}^{\mathrm{fin}}$. Therefore we pass to the completion C of the algebraic closure we have fixed in section 3.1. Consider a covering of the form $X_C \to X$ in $X_{\mathrm{pro\acute{e}t}}$; the construction is identical to the one in the proof of Lemma 3.2.3. On the localised site $X_{\mathrm{pro\acute{e}t}}/X_C$, we have an element t as in Definition 3.3.26. This depends on a choice of $\epsilon \in K^{\flat+}$. On $X_{\mathrm{pro\acute{e}t,affperfd}}^{\mathrm{fin}}/X_C$, we then get the presheaves

$$\begin{aligned} \mathbb{B}_{\mathrm{dR}}^{q,\mathrm{psh}} &: U \mapsto \mathbb{B}_{\mathrm{dR}}^{q}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right), \\ \mathbb{B}_{\mathrm{dR}}^{>q,\mathrm{psh}} &: U \mapsto \mathbb{B}_{\mathrm{dR}}^{>q}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right), \text{ and} \\ \mathbb{B}_{\mathrm{dR}}^{\infty,\mathrm{psh}} &: U \mapsto \mathbb{B}_{\mathrm{dR}}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right) \end{aligned}$$

of k_0-ind-Banach algebras. Write $\mathbb{B}_{\mathrm{dR}}^{\dagger,\mathrm{psh}} := \mathbb{B}_{\mathrm{dR}}^{\infty,\mathrm{psh}}$.

We view all the presheaves of Banach algebras above as presheaves of ind-Banach algebras, cf. Lemma 2.6.2. Now sheafification is allowed: Denote the sheafifications of all the presheaves above by $\mathbb{A}_{\mathrm{inf}}$, $\mathbb{A}_{\mathrm{dR}}^{q}$, $\mathbb{A}_{\mathrm{dR}}^{\dagger}$, $\widehat{\mathbb{B}}_{\mathrm{inf}}$, $\mathbb{B}_{\mathrm{dR}}^{q,+}$, $\mathbb{B}_{\mathrm{dR}}^{>q,+}$, $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$, $\mathbb{B}_{\mathrm{dR}}^{q}$, $\mathbb{B}_{\mathrm{dR}}^{>q}$, and $\mathbb{B}_{\mathrm{dR}}^{\dagger}$, for $q \in \mathbb{N} \cup \{\infty\}$, respectively. Since sheafification is strongly monoidal, see Lemma B.1.5, all of these sheaves are sheaves of $W(\kappa)$-ind-Banach algebras and k_0-ind-Banach algebras, respectively. By Lemma 3.2.6, these extend to sheaves of ind-Banach algebras on the pro-étale site $X_{\text{proét}}$.

Definition 3.4.1. $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ is the *positive overconvergent de Rham period sheaf.* $\mathbb{B}_{\mathrm{dR}}^{\dagger}$ is the *overconvergent de Rham period sheaf.*

Theorem 3.4.2. *Fix a symbol*

$$\mathbb{X} \in \left\{\mathbb{A}_{\mathrm{inf}}, \mathbb{A}_{\mathrm{dR}}^{>q}, \mathbb{A}_{\mathrm{dR}}^{\dagger}, \widehat{\mathbb{B}}_{\mathrm{inf}}, \mathbb{B}_{\mathrm{dR}}^{>q,+}, \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right\}$$

for $q \in \mathbb{N}_{\geq 2}$, together with an affinoid perfectoid $U \in X_{\text{proét}}$. Let $\widehat{U} = \mathrm{Spa}\,(R, R^+)$, where (R, R^+) denotes an affinoid perfectoid algebra over an affinoid perfectoid field (K, K^+). Then the canonical morphism

$$\mathbb{X}\left(R, R^+\right) \xrightarrow{\cong} \mathbb{X}(U)$$

is an isomorphism of $W(\kappa)$-ind-Banach-algebras and k_0-ind-Banach algebras, respectively. If K contains a compatible system of primitive pth roots of unity, then we get the following isomorphisms of k_0-ind-Banach algebras:

$$\begin{aligned} &\mathbb{B}_{\mathrm{dR}}^{>q}\left(R, R^+\right) \cong \mathbb{B}_{\mathrm{dR}}^{>q}(U) \quad \textit{and} \\ &\mathbb{B}_{\mathrm{dR}}^{\dagger}\left(R, R^+\right) \cong \mathbb{B}_{\mathrm{dR}}^{\dagger}(U). \end{aligned}$$

Remark 3.4.3. Theorem 3.4.2 also holds for $\mathbb{X} \in \left\{\mathbb{B}_{\mathrm{dR}}^{q,+}, \mathbb{B}_{\mathrm{dR}}^{q}\right\}$ and $q \in \mathbb{N}_{\geq 2}$. We do not know whether $\mathbb{A}_{\mathrm{dR}}^{q}(R, R^+) \to \mathbb{A}_{\mathrm{dR}}^{q}(U)$ is an isomorphism for finite q. However, we do know that it is an almost isomorphism, where the almost setup is chosen as in [54, Theorem 6.5]. We omit the proofs of these facts, as these results are not needed in the remainder of this text.

We need the following Lemma 3.4.4 in order to prove Theorem 3.4.2. All filtrations are descending.

Lemma 3.4.4. *Consider a strictly exact sequence*

$$M^{\bullet}\colon 0 \longrightarrow M' \xrightarrow{d'} M \xrightarrow{d} M''$$

of filtered abelian groups. Assume that $M''/\operatorname{Fil}^s M''$ has no p-power torsion for all $s \geq 0$. Equip each $N \in \{M', M, M''\}$ with the topology given by the open neighbourhood basis

$$\{p^s N + \operatorname{Fil}^s N\}_{s \geq 0}.$$

Then $M^\bullet$ is strictly exact as a complex of topological abelian groups.

Proof. Since $M^\bullet$ is strictly exact as a complex of filtered abelian groups,

$$M^{\bullet,s}\colon 0 \longrightarrow \frac{M'}{\operatorname{Fil}^s M'} \xrightarrow{d',s} \frac{M}{\operatorname{Fil}^s M} \xrightarrow{d^s} \frac{M''}{\operatorname{Fil}^s M''}$$

is exact for every $s \geq 0$. Now equip every $N^s \in \left\{\frac{M'}{\operatorname{Fil}^s M'}, \frac{M}{\operatorname{Fil}^s M}, \frac{M''}{\operatorname{Fil}^s M''}\right\}$ with the p-adic filtration, that is

$$\operatorname{Fil}^l N^s := p^l N^s \text{ for all } l \in \mathbb{N}.$$

Then $M^{\bullet,s}$ is strictly exact. This follows from the observation

$$d'^{,s}\left(p^l \frac{M'}{\operatorname{Fil}^s M'}\right) = p^l d'^{,s}\left(\frac{M'}{\operatorname{Fil}^s M'}\right) = p^l \ker d^s$$

for all $l \in \mathbb{N}$. The last step of this computation used that $M''/\operatorname{Fil}^s M''$ has no p-power torsion. This implies that the complexes

$$0 \longrightarrow \frac{M'}{\operatorname{Fil}^s M'}\Big/ p^s \longrightarrow \frac{M}{\operatorname{Fil}^s M}\Big/ p^s \longrightarrow \frac{M''}{\operatorname{Fil}^s M''}\Big/ p^s$$

are exact for all $s \geq 0$. But these complexes are isomorphic to

$$0 \longrightarrow \frac{M'}{p^s M' + \operatorname{Fil}^s M'} \longrightarrow \frac{M}{p^s M + \operatorname{Fil}^s M} \longrightarrow \frac{M''}{p^s M'' + \operatorname{Fil}^s M''}.$$

This implies

$$d'\left(p^s M' + \operatorname{Fil}^s M'\right) = d'\left(M'\right) \cap \left(p^s M + \operatorname{Fil}^s M\right).$$

That is, $M^\bullet$ is strictly exact as a complex of topological abelian groups. □

Proof of Theorem 3.4.2. It suffices to check

(i) $\mathbb{A}_{\text{inf}}(R, R^+) \xrightarrow{\cong} \mathbb{A}_{\text{inf}}(U)$,

(ii) $\mathbb{A}_{\text{dR}}^{>q}(R, R^+) \xrightarrow{\cong} \mathbb{A}_{\text{dR}}^{>q}(U)$,

and, if K contains a compatible system of pth roots of unity,

(iii) $\mathbb{B}_{\text{dR}}^{>q}(R, R^+) \xrightarrow{\cong} \mathbb{B}_{\text{dR}}^{>q}(U)$.

This follows from Corollary 2.2.2 and Lemma 3.3.15, as well as Lemma B.1.2, applied to the Corollaries 2.3.5 and 2.3.7. Note that these results apply, because $\mathbb{A}_{\text{inf}}(R, R^+)$ and $\mathbb{A}_{\text{dR}}^{>q}(R, R^+)$ are p-torsion free, cf. Lemma 3.3.1 and Corollary 3.3.19; here we are using that $q \geq 2$.

We check that $\mathbb{A}_{\text{inf}}^{\text{psh}}$, $\mathbb{A}_{\text{dR}}^{>q,\text{psh}}$, and $\mathbb{B}_{\text{dR}}^{>q,\text{psh}}$ are sheaves.

(i) By Lemma 2.6.2, it suffices to show that $\mathbb{A}_{\text{inf}}^{\text{psh}}$ is a sheaf of $W(\kappa)$-Banach algebras. We are working on the site $X_{\text{proét,affperfd}}^{\text{fin}}$, thus we have to show for every *finite* covering $\{U_i \to U\}_i$ that the complex

$$0 \longrightarrow \mathbb{A}_{\text{inf}}^{\text{psh}}(U) \longrightarrow \prod_i \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i) \longrightarrow \prod_{i,j} \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i \times_U U_j) \tag{3.4.1}$$

of $W(\kappa)$-Banach spaces is strictly exact [1]. Since the products are finite, both $\prod_i \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i)$ and $\prod_{i,j} \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i \times_U U_j)$ are again $W(\kappa)$-Banach spaces. In fact, they carry the $\left(p, \left[p^\flat\right]\right) = (p, \xi)$-adic topologies. Now Lemma 3.4.4 applies, because of Lemma 3.3.5(i), and it implies the following: It suffices to check that the complex (3.4.1) above is strictly exact, where the abelian groups $\mathbb{A}_{\text{inf}}^{\text{psh}}(U)$, $\prod_i \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i)$, and $\prod_{i,j} \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i \times_U U_j)$ carry the ξ-adic filtrations, that is

$$\text{Fil}^n := (\xi)^n \text{ for all } n \in \mathbb{N}.$$

This would follow once we computed that the associated graded complex is exact, cf. [38, Chapter I, subsection 4.2 page 31-32, Theorem 4(5)]. This is allowed because $\mathbb{A}_{\text{inf}}^{\text{psh}}(U)$, $\prod_i \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i)$, and $\prod_{i,j} \mathbb{A}_{\text{inf}}^{\text{psh}}(U_i \times_U U_j)$ are ξ-adically complete. Indeed, the products are finite and every $\mathbb{A}_{\text{inf}}^{\text{psh}}(V)$ is complete for every affinoid perfectoid V, cf. the proof of Lemma 3.3.3. By Lemma 3.3.2 and [28, Exercise 17.16.a], the associated graded of the complex (3.4.1) is

$$\begin{aligned} 0 \longrightarrow \widehat{\mathcal{O}}^+(U)[\sigma(\xi)] \longrightarrow & \prod_i \widehat{\mathcal{O}}^+(U_i)[\sigma(\xi)] \\ & \longrightarrow \prod_{i,j} \widehat{\mathcal{O}}^+(U_i \times_U U_j)[\sigma(\xi)], \end{aligned}$$

which is exact by [54, Lemma 4.10]; $\sigma(\xi)$ denotes the principal symbol of ξ.

The proof of (ii) is almost identical to the proof of (i).

[1] It has been shown in [54, Theorem 6.5(i)] that the underlying complex of abstract $W(\kappa)$-algebras is exact. However, the proof given *loc. cit.* does not establish strictness.

(ii) It suffices to show that $\mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}$ is a sheaf of $W(\kappa)$-Banach algebras, see Lemma 2.6.2. We are working on the site $X_{\mathrm{pro\acute{e}t,affperfd}}^{\mathrm{fin}}$, thus we have to show for every *finite* covering $\{U_i \to U\}_i$ that the complex

$$0 \longrightarrow \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U) \longrightarrow \prod_i \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i) \longrightarrow \prod_{i,j} \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i \times_U U_j) \tag{3.4.2}$$

of $W(\kappa)$-Banach spaces is strictly exact. Since the products are finite, both $\prod_i \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i)$ and $\prod_{i,j} \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i \times_U U_j)$ are again $W(\kappa)$-Banach spaces. In fact, they carry the $(p, \xi/p^q)$-adic topologies, Now Lemma 3.4.4 applies, because of Lemma 3.3.21, and it implies the following: It suffices to check that the complex (3.4.2) above is strictly exact, where the abelian groups $\mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U)$, $\prod_i \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i)$, and $\prod_{i,j} \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i \times_U U_j)$ carry the ξ/p^q-adic filtrations:

$$\mathrm{Fil}^n := \left(\frac{\xi}{p^q}\right)^n \text{ for all } n \in \mathbb{N}.$$

This would follow once we computed that the associated graded complex is exact, cf. [38, Chapter I, subsection 4.2 page 31-32, Theorem 4(5)]. This is allowed because $\mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U)$, $\prod_i \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i)$, and $\prod_{i,j} \mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(U_i \times_U U_j)$ are ξ/p^q-adically complete. Indeed, the products are finite and every $\mathbb{A}_{\mathrm{dR}}^{>q,\mathrm{psh}}(V)$ is complete for every affinoid perfectoid V, cf. [59, Tag 090T]. By Lemma 3.3.20 and [28, Exercise 17.16.a], the associated graded of the complex (3.4.1) is

$$0 \longrightarrow \widehat{\mathcal{O}}^+(U)\left[\sigma\left(\frac{\xi}{p^q}\right)\right] \longrightarrow \prod_i \widehat{\mathcal{O}}^+(U_i)\left[\sigma\left(\frac{\xi}{p^q}\right)\right] \\ \longrightarrow \prod_{i,j} \widehat{\mathcal{O}}^+(U_i \times_U U_j)\left[\sigma\left(\frac{\xi}{p^q}\right)\right],$$

which is exact by [54, Lemma 4.10]; $\sigma(\xi/p^q)$ denotes the principal symbol of ξ/p^q. The assumption $q \geq 2$ is used in the application of Lemma 3.3.18 and 3.3.20.

(iii) Let $\epsilon' \in K^{\flat +}$ denote a fixed compatible system of primity pth roots of unity. By Lemma 3.3.29, we can safely assume $\epsilon' = \epsilon$, the compatible system fixed earlier on in this subsection. We have also fixed the completion C of an algebraic closure of k, for which we fix an embedding $k \hookrightarrow C$. Furthermore, there is the canonical isomorphism of k_0-ind-Banach algebras

$$\begin{aligned} \mathbb{B}_{\mathrm{dR}}^{>q}(V) &\cong \text{“}\varinjlim_{t_K \times}\text{”}\, \mathbb{B}_{\mathrm{dR}}^{>q,+}(V) \\ &\cong \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(R, R^+\right) \widehat{\otimes}_{\mathbb{A}_{\mathrm{dR}}^1(C,C^+)} \text{“}\varinjlim_{t_K \times}\text{”}\, \mathbb{A}_{\mathrm{dR}}^1\left(C, C^+\right). \end{aligned} \tag{3.4.3}$$

for any affinoid perfectoid $V \in X_{\text{proét}}/X_C$; here we applied Corollary 2.3.5, using (ii) and the strict exactness of filtered colimits. Now compute, given the covering $U_C = U \times_{\mathrm{Spa}(K,K^+)} \mathrm{Spa}\left(C, C^+\right) \to U$,

$$\begin{aligned}
&\mathbb{B}_{\mathrm{dR}}^{>q}\left(U\right) \\
&\cong \ker\left(\mathbb{B}_{\mathrm{dR}}^{>q}\left(U_C\right) \to \mathbb{B}_{\mathrm{dR}}^{>q}\left(U_C \times_U U_C\right)\right) \\
&\overset{3.4.3}{\cong} \ker\begin{pmatrix} \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(U_C\right)\widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(C,C^+)}\text{“}\varinjlim_{t_K\times}\text{”}\mathbb{A}^1_{\mathrm{dR}}\left(C,C^+\right) \\ \to \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(U_C\times_U U_C\right)\widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(C,C^+)}\text{“}\varinjlim_{t_K\times}\text{”}\mathbb{A}^1_{\mathrm{dR}}\left(C,C^+\right)\end{pmatrix} \\
&\overset{3.3.30}{=} \ker\begin{pmatrix} \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(U_C\right)\widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)}\text{“}\varinjlim_{t_K\times}\text{”}\mathbb{A}^1_{\mathrm{dR}}\left(K,K^+\right) \\ \to \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(U_C\times_U U_C\right)\widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)}\text{“}\varinjlim_{t_K\times}\text{”}\mathbb{A}^1_{\mathrm{dR}}\left(K,K^+\right)\end{pmatrix} \\
&\cong \ker\left(\mathbb{B}_{\mathrm{dR}}^{>q,+}\left(U_C\right) \to \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(U_C\times_U U_C\right)\right)\widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)}\text{“}\varinjlim_{t_K\times}\text{”}\mathbb{A}^1_{\mathrm{dR}}\left(K,K^+\right) \\
&\cong \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(U\right)\widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)}\text{“}\varinjlim_{t_K\times}\text{”}\mathbb{A}^1_{\mathrm{dR}}\left(K,K^+\right) \\
&\overset{\text{(ii)}}{\cong} \mathbb{B}_{\mathrm{dR}}^{>q,+}\left(R,R^+\right)\widehat{\otimes}_{\mathbb{A}^1_{\mathrm{dR}}(K,K^+)}\text{“}\varinjlim_{t_K\times}\text{”}\mathbb{A}^1_{\mathrm{dR}}\left(K,K^+\right) \\
&= \mathbb{B}_{\mathrm{dR}}^{>q}\left(R,R^+\right)
\end{aligned}$$

This finishes the proof of Theorem 3.4.2. □

3.5 The overconvergent de Rham period structure sheaf

Keep the notation from subsection 3.4 fixed. X denotes again a locally Noetherian adic space over $\mathrm{Spa}(k, k^\circ)$.

For every $q \in \mathbb{N} \cup \{\infty\}$, we introduce the presheaves k-ind-Banach algebras

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+,\mathrm{psh}} \colon U = \text{“}\varprojlim_{i\in I}\text{”}U \mapsto \text{“}\varinjlim_{i\in I}\text{”}\left(\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U)\right)\left\langle\frac{\ker\mathcal{O}\theta_{\mathrm{inf}}}{p^q}\right\rangle\widehat{\otimes}_{k^\circ}k$$

on $X^{\mathrm{fin}}_{\text{proét,affperfd}}$. Here, $\mathcal{O}\theta_{\mathrm{inf}}$ denotes the composition of the surjections

$$\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U) \xrightarrow{\mathrm{id}\widehat{\otimes}\theta_{\mathrm{inf}}} \mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\widehat{\mathcal{O}}^+(U) \xrightarrow{\mu^+} \widehat{\mathcal{O}}^+(U),$$

where μ^+ denotes the multiplication. $\mathcal{O}^+(U_i)$ carries the π-adic seminorm.

Remark 3.5.1. The kernels of the maps $\mathcal{O}\theta_{\mathrm{inf}}$ are finitely generated. This follows from Lemma 3.3.10 and because the kernel of μ^+ is finitely generated. Now Lemma 2.5.10 allows to give a more concrete definition of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+,\mathrm{psh}}(U)$.

Since sheafification is strongly monoidal, cf. Lemma B.1.5, the sheafifications $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+}$ of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+,\mathrm{psh}}$ are sheaves of k-ind-Banach algebras. They extend, by Lemma 3.2.6, to sheaves on the whole pro-étale site, which we denote again by $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+}$.

Use Lemma 2.6.2 to view the structure sheaf $\mathcal{O}$ on X as a sheaf of k-ind-Banach algebras. The canonical morphisms

$$\nu^{-1}\mathcal{O}\widehat{\otimes}_{k_0}\mathbb{B}_{\mathrm{dR}}^{q,+} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+}$$

are morphisms of sheaves of k-ind-Banach algebras, cf. Lemma B.1.5. This makes $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+}$ a sheaf of $\nu^{-1}\mathcal{O}$-ind-Banach algebras, and a sheaf of $\mathbb{B}_{\mathrm{dR}}^{q,+}$-ind-Banach algebras.

The maps $\mathcal{O}\theta_{\mathrm{inf}}$ induce morphisms

$$\mathcal{O}\vartheta_{\mathrm{dR}}^{q,+}\colon\ \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+} \to \widehat{\mathcal{O}}$$

of sheaves of k-ind-Banach spaces for every $q \in \mathbb{N}\cup\{\infty\}$, by Lemma 2.5.9 and B.1.5.

Definition 3.5.2. $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} := \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\infty,+}$ is the *positive overconvergent de Rham period structure sheaf.*

Again, $q \in \mathbb{N}\cup\{\infty\}$. C denotes the completion of an algebraic extension of k that we have fixed in of section 3.1. $C^+ \subseteq C$ is a fixed ring of integral elements. The element $t \in A_{\mathrm{dR}}^1 := \mathbb{A}_{\mathrm{dR}}^1(C, C^+)$ has been introduced in Definition 3.3.26. Recall Notation 3.3.27 and define the following sheaf of k-ind-Banach spaces on $X_{\mathrm{pro\acute{e}t}}/X_C$:

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q} := \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q,+}\widehat{\otimes}_{A_{\mathrm{dR}}^1}\text{“}\varinjlim_{t\times}\text{”}A_{\mathrm{dR}}^1.$$

By Lemma 3.2.6, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q}$ extends to a sheaf on the pro-étale site, which we denote again by $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{q}$.

Definition 3.5.3. $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger} := \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\infty}$ is the *overconvergent de Rham period structure sheaf.*

We aim to describe the period structure sheaves locally, in the spirit of [54, Proposition 6.10]. Assume that X is affinoid and equipped with an étale map $X \to \mathbb{T}^d := \mathbb{T}_0^d$. Here we define, for all $e \in \mathbb{N}$,

$$\mathbb{T}_e^d := \mathrm{Spa}\left(k\left\langle T_1^{\pm 1/p^e}, \dots, T_d^{\pm 1/p^e}\right\rangle, k^\circ\left\langle T_1^{\pm 1/p^e}, \dots, T_d^{\pm 1/p^e}\right\rangle\right)$$

Write $\widetilde{\mathbb{T}}^d := \text{“}\varprojlim_{e\in\mathbb{N}}\text{”}\, \mathbb{T}^d_e \in \mathbb{T}^d_{\text{proét}}$ and $\widetilde{X} := X \times_{\mathbb{T}^d} \widetilde{\mathbb{T}}^d$. Every $U = \text{“}\varprojlim_{i\in I}\text{”} U_i \in X_{\text{proét}}/\widetilde{X}$ gives rise to a morphism $U \to \widetilde{X} \to \widetilde{\mathbb{T}}^d$ in $\mathbf{Pro}\left(\mathbb{T}^d_{\text{proét}}\right)$. It is thus given by a compatible system of étale maps $U_{i_0} \to \mathbb{T}^d_e$ for a fixed $i_0 \in I$ and varying e.

Notation 3.5.4. Fix such an $i_0 \in I$.

Consider the images of the elements $T_1, \dots, T_d \in \mathcal{O}^+\left(\mathbb{T}^d_e\right)$ in $\mathcal{O}^+(U_i)$ for all $i \geq i_0$ and e large enough. Abusing notation, denote them again by $T_1, \dots, T_d$.

$$u_l := T_l \widehat{\otimes} 1 - 1 \widehat{\otimes} \left[T_l^\flat\right] \in \mathcal{O}^+(U_i) \widehat{\otimes}_{W(\kappa)} \mathbb{A}_{\text{inf}}(U) \tag{3.5.1}$$

for all $l = 1, \dots, d$, every $U = \text{“}\varprojlim_{i\in I}\text{”} U_i \in X_{\text{proét,affperfd}}/\widetilde{X}$, and $i \geq i_0$. Here

$$T_l^\flat := \left(T_l, T_l^{1/p}, T_l^{1/p^2}, \dots\right) \in \left(\widehat{\mathcal{O}}^+(U)\right)^\flat.$$

Since those u_l lie in the kernel of $\mathcal{O}\theta_{\text{inf}}$, the canonical maps

$$\mathbb{A}^q_{\text{dR}}(U) \to \left(\mathcal{O}^+(U_i) \widehat{\otimes}_{W(\kappa)} \mathbb{A}_{\text{inf}}(U)\right) \left\langle \frac{\ker \vartheta}{p^q} \right\rangle$$

extend with Corollary 2.4.11 to the morphisms

$$\mathbb{A}^q_{\text{dR}}(U) \left\langle \frac{Z_1, \dots, Z_d}{p^q} \right\rangle \to \left(\mathcal{O}^+(U_i) \widehat{\otimes}_{W(\kappa)} \mathbb{A}_{\text{inf}}(U)\right) \left\langle \frac{\ker \vartheta}{p^q} \right\rangle, Z_l/p^q \mapsto u_l/p^q$$

of $\mathbb{A}^q_{\text{dR}}(U)$-Banach algebras for every $q \in \mathbb{N}$. Here, the Z_l denote formal variables [2]. Invert p and pass to the colimit along $i \in I$ to get

$$\Phi^{q,+}(U) \colon \mathbb{B}^{q,+}_{\text{dR}}(U) \left\langle \frac{Z_1, \dots, Z_d}{p^q} \right\rangle \to \mathcal{O}\mathbb{B}^{q,+,\text{psh}}_{\text{dR}}(U).$$

The data of these maps $\Phi^{q,+}_{\text{dR}}(U)$ define morphisms

$$\Phi^{q,+,\text{psh}} \colon \mathbb{B}^{q,+}_{\text{dR}}|_{\widetilde{X}} \left\langle \frac{Z_1, \dots, Z_d}{p^q} \right\rangle^{\text{psh}} \to \mathcal{O}\mathbb{B}^{q,+,\text{psh}}_{\text{dR}}|_{\widetilde{X}}$$

of presheaves of $\mathbb{B}^{q,+}_{\text{dR}}|_{\widetilde{X}}$-ind-Banach algebras on $X_{\text{proét}}/\widetilde{X}$, where

$$\mathbb{B}^{q,+}_{\text{dR}}|_{\widetilde{X}} \left\langle \frac{Z_1, \dots, Z_d}{p^q} \right\rangle^{\text{psh}} : V \mapsto \mathbb{B}^{q,+}_{\text{dR}}(V) \left\langle \frac{Z_1, \dots, Z_d}{p^q} \right\rangle.$$

By Lemma B.1.5, its sheafification

$$\Phi^{q,+} \colon \mathbb{B}^{q,+}_{\text{dR}}|_{\widetilde{X}} \left\langle \frac{Z_1, \dots, Z_d}{p^q} \right\rangle \to \mathcal{O}\mathbb{B}^{q,+}_{\text{dR}}|_{\widetilde{X}}$$

[2] [54] denotes Z_l as X_l. We use different symbols to avoid confusion with the space X.

is a morphism of sheaves of $\mathbb{B}_{\mathrm{dR}}^{q,+}|_{\widetilde{X}}$-ind-Banach algebras. Here,

$$\mathbb{B}_{\mathrm{dR}}^{q,+}|_{\widetilde{X}}\left\langle\frac{Z_1,\dots,Z_d}{p^q}\right\rangle := \left(\mathbb{B}_{\mathrm{dR}}^{q,+}|_{\widetilde{X}}\left\langle\frac{Z_1,\dots,Z_d}{p^q}\right\rangle^{\mathrm{psh}}\right)^{\mathrm{sh}}.$$

Now pass to the colimit along $q\to\infty$ to obtain the morphism

$$\Phi^{\dagger,+}\colon\ \mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}}\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}},$$

of sheaves of $\mathbb{B}_{\mathrm{dR}}^{+}|_{\widetilde{X}}$-ind-Banach algebras. Finally, using the notation in the paragraph following Definition 3.5.2,

$$\Phi_C^\dagger := \left(\Phi^{\dagger,+}|_{\widetilde{X}_C}\widehat{\otimes}_{A^1_{\mathrm{dR}}}\mathrm{id}_{\text{“}\varinjlim_{t\times}\text{”}A^1_{\mathrm{dR}}}\right)^{\mathrm{sh}}.$$

This is a morphism of sheaves of $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}$-ind-Banach algebras on $X_{\text{proét}}/\widetilde{X}_C$, cf. Lemma B.1.5. By Lemma 3.2.6, there exists a unique morphism $\Phi^\dagger$ with $\Phi^\dagger|_{\widetilde{X}_C}=\Phi_C^\dagger$. Write

$$\Phi^{\dagger}\colon\ \mathbb{B}_{\mathrm{dR}}^{\dagger}|_{\widetilde{X}}\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger}|_{\widetilde{X}}.$$

$\mathbb{B}_{\mathrm{dR}}^{\dagger}|_{\widetilde{X}}\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle$ is the sheafification of the presheaf whose restriction to $\widetilde{X}_C$ is

$$U\mapsto \mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle\widehat{\otimes}_{A^1_{\mathrm{dR}}}\text{“}\varinjlim_{t\times}\text{”}A^1_{\mathrm{dR}}\cong \mathbb{B}_{\mathrm{dR}}^{\dagger}(U)\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle.$$

Theorem 3.5.5. *Let X be affinoid and equipped with an étale map $X\to\mathbb{T}^d$, giving rise to the pro-étale covering $\widetilde{X}\to X$. Then the morphism*

$$\Phi^{\dagger,+}\colon\ \mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}}\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle \xrightarrow{\cong} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}}$$

is an isomorphism of sheaves of $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}|_{\widetilde{X}}$-ind-Banach algebras, and

$$\Phi^{\dagger}\colon\ \mathbb{B}_{\mathrm{dR}}^{\dagger}|_{\widetilde{X}}\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle \xrightarrow{\cong} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger}|_{\widetilde{X}}$$

is an isomorphism of sheaves of $\mathbb{B}_{\mathrm{dR}}^{\dagger}|_{\widetilde{X}}$-ind-Banach algebras.

Remark 3.5.6. Theorem 3.5.5 implies its version [54, Proposition 6.10] for $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$ via taking t-adic completions, locally on the pro-étale site.

We prove Theorem 3.5.5 in subsection 3.6.

Corollary 3.5.7. *Let X be affinoid and equipped with an étale map $X \to \mathbb{T}^d$, giving rise to the pro-étale covering $\widetilde{X} \to X$. The morphism*

$$\begin{aligned} \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(R, R^+\right)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle &\xrightarrow{\cong} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U) \\ Z_l &\longmapsto u_l \end{aligned}$$

of $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(R,R^+)$-ind-Banach algebras is an isomorphism for any affinoid perfectoid $U \in X_{pro\acute{e}t}/\widetilde{X}$ with $\widehat{U} = \operatorname{Spa}(R,R^+)$. If further $U \in X_{pro\acute{e}t}/\widetilde{X}_K$ for a perfectoid field K containing a compatible sequence of primitive pth roots of unity, then

$$\begin{aligned} \mathbb{B}_{\mathrm{dR}}^{\dagger}\left(R, R^+\right)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle &\xrightarrow{\cong} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger}(U), \\ Z_l &\longmapsto u_l. \end{aligned}$$

Proof. It suffices to check that

$$U \mapsto \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\right)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle \tag{3.5.2}$$

is a sheaf on $X^{\mathrm{fin}}_{\mathrm{pro\acute{e}t,affperfd}}$. Indeed, then we can apply the Theorems 3.4.2 and 3.5.5. But the sheafiness of (3.5.2) follows from Theorem 3.4.2 together with Lemma B.1.2, which applies because of Corollary 2.4.16. □

Remark 3.5.8. Write $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+,\mathrm{psh}} := \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\infty,+,\mathrm{psh}}$. Then, for any $U \in X_{\mathrm{pro\acute{e}t}}/\widetilde{X}$,

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+,\mathrm{psh}}(U) \cong \mathbb{B}_{\mathrm{dR}}^{\dagger,\mathrm{psh}}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle \cong \mathbb{B}_{\mathrm{dR}}^{\dagger}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle \cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$$

by the proofs of Theorem 3.5.5 and Corollary 3.5.7.

3.6 Proof of Theorem 3.5.5

Fix the setup as described in Theorem 3.5.5. It suffices to check that $\Phi_{\mathrm{dR}}^{\dagger,+}$ is an isomorphism. We may show that the maps

$$\Phi_{\mathrm{dR}}^{\dagger,+,\mathrm{psh}}(U) : \mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+,\mathrm{psh}}(U)$$

are isomorphisms for every affinoid perfectoid $U \in X_{\mathrm{pro\acute{e}t}}/\widetilde{X}$. Fix such a U with $\widehat{U} = \operatorname{Spa}(R,R^+)$, together with a pro-étale presentation $U = “\varprojlim_{i\in I}” U_i \in X_{\mathrm{pro\acute{e}t}}/\widetilde{X}$, $U_i = \operatorname{Spa}\left(R_i, R_i^+\right)$ for all $i \in I$. Let $i \geq i_0$ be arbitrary, cf. Notation 3.5.4. We will show that the morphism

$$\phi_i \colon \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(R, R^+\right)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle \to \left(\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U)\right)\left\langle\frac{\ker \mathcal{O}\theta_{\mathrm{inf}}}{p^\infty}\right\rangle\widehat{\otimes}_{k^\circ} k \tag{3.6.1}$$

is an isomorphism. This suffices because $\Phi_{\mathrm{dR}}^{\dagger,+,\mathrm{psh}}(U) = \varinjlim_{i\geq i_0} \phi_i$.

Lemma 3.6.1. *The assignment $T_l \mapsto [T_l^\flat] + Z_l$ defines a unique morphism*

$$\widetilde{\epsilon}_i^+ \colon R_i^+ \to \mathbb{A}_{\mathrm{inf}}(R, R^+)[\![Z_1, \ldots, Z_d]\!].$$

of $W(\kappa)$-algebras. It fits into the commutative diagram

$$\begin{array}{ccc} R_i^+ & \xrightarrow{\widetilde{\epsilon}_i^+} & \mathbb{A}_{\mathrm{inf}}(R, R^+)[\![X_1, \ldots, X_d]\!] \\ & \searrow & \Big\downarrow \mathcal{O}\theta'_{\mathrm{inf}} \\ & & R^+, \end{array} \tag{3.6.2}$$

where $\mathcal{O}\theta'_{\mathrm{inf}}$ is the map $\sum_{\alpha \in \mathbb{N}^d} a_\alpha X^\alpha \mapsto \theta(a_0)$.

In order to prove Lemma 3.6.1, we cite the following multidimensional version of Hensel's Lemma from [24, Corollary 4.5.2].

Lemma 3.6.2. *Fix a commutative, linearly topologised, Hausdorff and complete ring A, as well as a closed ideal $\mathfrak{m} \subseteq A$, whose elements are topologically nilpotent. Let $f = (f_1, \ldots, f_n)$ denote a tuple of polyomials in $A[L]$ in variables $L = (L_1, \ldots, L_n)$ and let $J_f(L) \in A[L]$ denote the Jacobian, that is the determinant of the matrix*

$$\left(\frac{\partial f}{\partial L}\right) := \left(\frac{\partial f_i}{\partial L_j}\right)_{i,j=1,\ldots,n}.$$

Consider $a \in A^n$ such that $J_f(a)$ is invertible in A and $f_i(a) \equiv 0 \mod \mathfrak{m}$ for all $i = 1, \ldots, n$. Then there exists a unique $x \in A^n$ such that $x_i \equiv a_i \mod \mathfrak{m}$ for all i and $f_i(x) = 0$ for all $i = 1, \ldots, n$.

We closely follow the [54, proof of Proposition 6.10].

Proof of Lemma 3.6.1. Pick a finitely generated $W(\kappa)\left[T_1^{\pm 1}, \ldots, T_d^{\pm 1}\right]$-algebra R_{i0}^+ whose p-adic completion is R_i^+ such that $R_{i0} = R_{i0}^+[1/p]$ is étale over $W(\kappa)[1/p]\left[T_1^{\pm 1}, \ldots, T_d^{\pm 1}\right]$, see [54, Lemma 6.12]. This gives a finite presentation

$$R_{i0} = \left(W(\kappa)[1/p]\left[T_1^{\pm 1}, \ldots, T_d^{\pm 1}\right]\right)[L_{i1}, \ldots, L_{in_i}] / (P_{i1}, \ldots, P_{in_i})$$

such that the Jacobian $J_P(L)$ is a unit in R_{i0}. Here we used multi-index notation $L := (L_{i1}, \ldots, L_{in_i})$ and $P := (P_{i1}, \ldots, P_{in_i})$, omitting the index i for clarity. Without loss of generality, we can assume that

(i) $L_{ij} \in R_{i0}^+$ for every $j = 1, \ldots, n_i$,

(ii) $P_{ij} \in W(\kappa)\left[T_1^{\pm 1}, \ldots, T_d^{\pm 1}\right][L_{i1}, \ldots, L_{in_i}]$ for every $j = 1, \ldots, n_i$, and

(iii) the Jacobian $J_P(L) \in R_{i0}^+$ is invertible.

Here is why:

(i) Because $R_{i0} = R_{i0}^+[1/p]$, there exists an $s \in \mathbb{N}$ such that $L_{ij}^+ := p^s L_{ij} \in R_{i0}^+$ for all $j = 1, \ldots, n_i$. Indeed, we have

$$R_{i0} = \left(W(\kappa)[1/p]\left[T_1^{\pm 1}, \ldots, T_d^{\pm 1}\right]\right)\left[L_{i1}^+, \ldots, L_{in_i}^+\right] / \left(P_{i1}, \ldots, P_{in_i}\right).$$

Write $L^+ := \left(L_{i1}^+, \ldots, L_{in_i}^+\right)$, omitting again the subscript i. We claim that the Jacobian $J_P(L^+)$ is still a unit in R_{i0}. The chain rule yields

$$\begin{aligned} J_P(L) = \det\left(\left(\frac{\partial P}{\partial L}\right)\right) &= \det\left(\left(\frac{\partial P}{\partial L^+}\right) \cdot \left(\frac{\partial L^+}{\partial L}\right)\right) \\ &= \det\left(\left(\frac{\partial P}{\partial L^+}\right)\right)\det\left(\left(\frac{\partial L^+}{\partial L}\right)\right). \end{aligned}$$

$J_P(L)$ is a unit by assumption and

$$\det\left(\left(\frac{\partial L^+}{\partial L}\right)\right) = \det\left(p^s I_{n_i}\right) = p^{n_i s}$$

is a unit in R_{i0}. Thus $\det\left(\left(\frac{\partial P}{\partial L^+}\right)\right)$ is a unit in R_{i0} too, as desired.

(ii) We have to multiply each P_{ij} by a suitable power of p. Note that this multiplies $J_P(L)$ by a power of p, such that the Jacobian is still a unit in R_{i0}.

(iii) Multiply each P_{ij} with an appropriate power of p such that $J(L)$ becomes invertible in R_{i0}^+. Note that this does not affect the assumptions (i) and (ii) above.

Assume (i), (ii), and (iii) without loss of generality. The $\left[T_l^\flat\right] + Z_l$ are invertible in $\mathbb{A}_{\inf}(R, R^+)[\![Z_1, \ldots, Z_d]\!]$, with inverses $\sum_{n \geq 0}(-1)^n \left[T_l^\flat\right]^{-n-1} Z_l^n$. Therefore, we have a morphism

$$W(\kappa)\left[T_1^{\pm 1}, \ldots T_d^{\pm 1}\right] \to \mathbb{A}_{\inf}\left(R, R^+\right)[\![Z_1, \ldots, Z_d]\!] \tag{3.6.3}$$

of $W(\kappa)$-algebras sending T_l to $\left[T_l^\flat\right] + Z_l$ for all $l = 1, \ldots, d$. Let P'_{ij} denote the image of the polynomial P_{ij} under the map

$$W(\kappa)[T_1^{\pm 1}, \ldots, T_d^{\pm 1}][L_{i1}, \ldots, L_{in_i}] \to \mathbb{A}_{\inf}(R, R^+)[\![Z_1, \ldots, Z_d]\!][L_{i1}, \ldots, L_{in_i}]$$

induced by 3.6.3. We summarise the setup and compare it to Lemma 3.6.2.

- Equip $A := \mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!]$ with the $(p, \xi, Z_1, \dots, Z_d)$-adic topology. Apply Proposition C.0.1 to Corollary 3.3.3 and find that this defines a commutative, linearly topologised, Hausdorff, and complete ring.

- The ideal $\mathfrak{m} := (\xi, Z_1, \dots, Z_d)$ is closed. Indeed, it is the kernel of the map

$$A = \mathbb{A}_{\inf}\left(R, R^+\right)[\![Z_1, \dots, Z_d]\!] \to R^+, \sum_{\alpha \in \mathbb{N}^d} a_\alpha Z^\alpha \mapsto \theta_{\inf}(a_0),$$

which is continuous. Here, R^+ carries the p-adic topology. The elements of $\mathfrak{m}$ are topologically nilpotent because its generators are.

- We have polynomials $P'_{i1}, \dots, P'_{in_i}$ in $\mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!][L_{i1}, \dots, L_{in_i}]$. They correspond to the elements f_j in Lemma 3.6.2.

- We have $\mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!]/\mathfrak{m} \xrightarrow{\cong} R^+$. Since $R^+_{i0} \subseteq R^+$, one can pick lifts $L'_{ij} \in \mathbb{A}_{\inf}(R, R^+)[\![X_1, \dots, X_d]\!]$ of the elements $L_{i1}, \dots, L_{in} \in R^+_{i0}$. Write $L' = (L'_{i1}, \dots, L'_{in})$, omitting the index i for clarity. Then the Jacobian $J_{P'}(L')$ is a lift of $J_P(L) \in R^+_{i0}$. Since the latter is a unit, [59, Tag 05GI] implies that the lift is a unit. Here we have used that $\mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!]$ is complete with respect to the $\mathfrak{m}$-adic topology. This follows from Lemma [59, Tag 090T], because $\mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!]$ is complete with respect to the $(p, \xi, Z_1, \dots, Z_d)$-adic topology, as explained above.

- We have $P'^{+}_{ij}(L'^{+}_{ij}) \equiv P_{ij}(L_{ij}) \equiv 0$ modulo $\mathfrak{m}$ for every $j = 1, \dots, n_i$.

Thus one can apply Hensel's Lemma 3.6.2. We find a unique tuple $\widetilde{L} = \left(\widetilde{L}_{i1}, \dots, \widetilde{L}_{in}\right)$ with entries in $\mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!]$ such that $P_{ij}\left(\widetilde{L}_{ij}\right) = 0$ and $\widetilde{L}_{ij} \equiv L'_{ij}$ modulo $\mathfrak{m}$ for all $j = 1, \dots n_i$. Now define the map

$$\widetilde{\epsilon}^+_{i0} \colon R^+_{i0} \to \mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!]$$

of $W(\kappa)\left[T_1^{\pm}, \dots, T_d^{\pm}\right]$-algebras by $L_{ij} \mapsto \widetilde{L}_{ij}$ for all $j = 1, \dots, n_i$. $\mathbb{A}_{\inf}(R, R^+)$ is p-adically complete by Lemma 3.3.1, thus $\mathbb{A}_{\inf}(R, R^+)[\![Z_1, \dots, Z_d]\!]$ is p-adically complete by Lemma C.0.3. Therefore, $\widetilde{\epsilon}^+_{i0}$ extends to the desired morphism $\widetilde{\epsilon}^+_i$ by continuity. Here we have used that R^+_i is p-adically complete, which can be seen with arguments similar to the ones at the [12, end of the proof of Lemma 3.6.1].

The uniqueness follows from the following: $\widetilde{\epsilon}^+_i$ is determined by its restriction to R^+_{i0}, and this restriction is unique by Hensel's Lemma 3.6.2.

Finally, we get the commutative diagram (3.6.2) from the computations

$$\mathcal{O}\theta'_{\mathrm{inf}}(\widetilde{\epsilon}_i^+(L_{ij})) = \mathcal{O}\theta'_{\mathrm{inf}}\left(\widetilde{L}_{ij}\right) = \mathcal{O}\theta'_{\mathrm{inf}}\left(L'_{ij}\right) = L_{ij}$$

for all $j = 1, \dots, n_i$. □

Notation 3.6.3. Introduce the following bounded linear map

$$\begin{aligned} \epsilon_i^+ \colon R_i^+ \xrightarrow{\widetilde{\epsilon}_i^+} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right)[\![Z_1, \dots, Z_d]\!] &\longrightarrow \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right) \\ \sum_{\alpha \in \mathbb{N}^d} a_\alpha Z^\alpha &\longmapsto a_0. \end{aligned}$$

The map ϵ_i^+ gives $\mathbb{A}_{\mathrm{inf}}\left(R, R^+\right)$ the structure of an R_i^+-Banach algebra and $\epsilon_i := \epsilon_i^+ \widehat{\otimes}_{W(\kappa)} k_0$ makes $\widehat{\mathbb{B}}_{\mathrm{inf}}\left(R, R^+\right)$ an R_i-Banach algebra. We want exhibit ϕ_i as a base-change of a certain τ_i along ϵ_i, following the proof of the local description of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ [55].

Notation 3.6.4. Denote the multiplication $R_i^+ \widehat{\otimes}_{W(\kappa)} R_i^+ \to R_i^+$ by μ_i^+.

Lemma 3.6.5.

$$G^+ := \left(\mathrm{id}_{R_i^+} \widehat{\otimes}_{W(\kappa)} \epsilon_i^+\right)\left(\ker \mu_i^+\right) \cup \left\{1 \widehat{\otimes} \xi\right\} \subseteq R_i^+ \widehat{\otimes}_{W(\kappa)} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right)$$

generates the ideal $\ker \mathcal{O}\theta_{\mathrm{inf}}$.

Proof. Drop the subscripts $W(\kappa)$ for clarity. $\mathcal{O}\theta_{\mathrm{inf}}$ is the composition

$$R_i^+ \widehat{\otimes} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right) \xrightarrow{\mathrm{id} \widehat{\otimes} \theta_{\mathrm{inf}}} R_i^+ \widehat{\otimes} R^+ \xrightarrow{\mu^+} R^+,$$

where μ^+ is the multiplication. Clearly, $1 \widehat{\otimes} \xi \in \ker \mathcal{O}\theta_{\mathrm{inf}}$. For every $g \in \ker \mu_i^+$,

$$\begin{aligned} \mathcal{O}\theta_{\mathrm{inf}}\left(\left(\mathrm{id}_{R_i^+} \widehat{\otimes} \epsilon_i^+\right)(g)\right) &= \left(\mu \circ \left(\mathrm{id}_{R_i^+} \widehat{\otimes} \theta_{\mathrm{inf}}\right) \circ \left(\mathrm{id}_{R_i^+} \widehat{\otimes} \epsilon_i^+\right)\right)(g) \\ &= \left(\mu \circ \left(\mathrm{id}_{R_i^+} \widehat{\otimes} \left(\theta_{\mathrm{inf}} \circ \epsilon_i^+\right)\right)\right)(g) \\ &\overset{3.6.1}{=} \left(\mu \circ \left(\mathrm{id}_{R_i^+} \widehat{\otimes} \iota_i^+\right)\right)(g) \\ &= 0, \end{aligned} \tag{3.6.4}$$

where $\iota_i^+ \colon R_i^+ \to R^+$ is the canonical map. This proves $(G) \subseteq \ker \mathcal{O}\theta_{\mathrm{inf}}$. We aim to show $\supseteq$ via computing that $\mathcal{O}\theta_{\mathrm{inf}}$ induces an isomorphism

$$\left(R_i^+ \widehat{\otimes} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right)\right) / (G) \xrightarrow{\cong} R^+.$$

By the third isomorphism theorem, this breaks into two parts. First,

$$\left(R_i^+ \widehat{\otimes} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right)\right) / \left(1 \widehat{\otimes} \xi\right) \xrightarrow{\cong} R_i^+ \widehat{\otimes} R^+; \tag{3.6.5}$$

this isomorphism is induced by $\mathrm{id}_{R_i^+} \widehat{\otimes} \mathcal{O}\theta_{\mathrm{inf}}$. Second, μ^+ induces

$$\left(R_i^+\widehat{\otimes}R^+\right)\big/\left(\left(\mathrm{id}_{R_i^+}\widehat{\otimes}\theta_{\mathrm{inf}}\right)\left(\left(\mathrm{id}_{R_i^+}\widehat{\otimes}\epsilon_i^+\right)\left(\ker\mu_i^+\right)\right)\right) \xrightarrow{\cong} R^+. \tag{3.6.6}$$

The first isomorphism comes from the strictly coexact sequence

$$\mathbb{A}_{\mathrm{inf}}\left(R, R^+\right) \xrightarrow{\xi} \mathbb{A}_{\mathrm{inf}}\left(R, R^+\right) \longrightarrow R^+ \longrightarrow 0,$$

cf. Lemma 3.3.2 and Corollary 3.3.4. Indeed, applying $R_i^+\widehat{\otimes}_{W(\kappa)}$ gives the sequence

$$R_i^+\widehat{\otimes}\mathbb{A}_{\mathrm{inf}}\left(R, R^+\right) \xrightarrow{\xi} R_i^+\widehat{\otimes}\mathbb{A}_{\mathrm{inf}}\left(R, R^+\right) \longrightarrow R_i^+\widehat{\otimes}R^+ \longrightarrow 0,$$

which is strictly coexact. This gives (3.6.5). Regarding the second isomorphism (3.6.6),

$$\left(\left(\mathrm{id}_{R_i^+}\widehat{\otimes}\theta_{\mathrm{inf}}\right)\left(\left(\mathrm{id}_{R_i^+}\widehat{\otimes}\epsilon_i^+\right)\left(\ker\mu_i^+\right)\right)\right) = \left(\mathrm{id}_{R_i^+}\widehat{\otimes}\iota_i^+\right)\left(\ker\mu_i^+\right),$$

by the computation 3.6.4 above. We may thus write (3.6.6) as

$$\left(R_i^+\widehat{\otimes}R^+\right)\big/\left(\left(\mathrm{id}_{R_i^+}\widehat{\otimes}\iota_i^+\right)\left(\ker\mu_i^+\right)\right) \xrightarrow{\cong} R^+. \tag{3.6.7}$$

We claim that the image of $\ker\mu_i^+ \subseteq A := R_i^+\widehat{\otimes}R_i^+$ in $B := R_i^+\widehat{\otimes}R^+$ generates the kernel of the multiplication map as an ideal. Apply [2, Theorem 4.1.4] to get an exact sequence of finitely presented A-modules

$$A^n \longrightarrow A \xrightarrow{\mu_i^+} R_i^+ \longrightarrow 0.$$

Apply the functor $-\widehat{\otimes}_A B$ to get an exact sequence of finitely presented B-modules

$$B^n \longrightarrow B \xrightarrow{\mu_i^+\widehat{\otimes}_A \mathrm{id}_B} R_i^+\widehat{\otimes}_A B \longrightarrow 0.$$

The functors $-\widehat{\otimes}_{R_i^+}R$ and $-\widehat{\otimes}_A B$ are isomorphic on finitely generated A-modules. This is because both are suitably right exact, and both send A to B, up to natural isomorphism. Hence we get the exact sequence

$$B^n \longrightarrow B \xrightarrow{\upsilon^+} R_i^+\widehat{\otimes}_{R_i^+}R^+ \cong R^+ \longrightarrow 0.$$

One the other hand, $\mu^+\colon B \to R^+$ kills the image of A^n in B, giving a complex

$$B^n \longrightarrow B \xrightarrow{\mu^+} R^+ \longrightarrow 0.$$

One checks that this complex is equal to the previous exact sequence: the only non-trivial part is to show that μ^+ and υ^+ coincide. But υ^+ is the composition

$$\begin{array}{ccccccc} R_i^+\widehat{\otimes}R^+ = B & \xrightarrow{\cong} & A\widehat{\otimes}_A B & \xrightarrow{\mu_i^+\widehat{\otimes}_A B} & R_i^+\widehat{\otimes}_A B & \xrightarrow{\cong} & R_i^+ \\ r\widehat{\otimes}s & \longmapsto & \left(r\widehat{\otimes}1\right)\widehat{\otimes}\left(1\widehat{\otimes}s\right) & \longmapsto & r\widehat{\otimes}\left(1\widehat{\otimes}s\right) & \longmapsto & rs, \end{array}$$

which shows the claim. It follows that the image of $\ker\mu_i^+ \subseteq A$ in B generates the multiplication map. This gives (3.6.7). □

We use Lemma 3.6.6 without further reference.

Lemma 3.6.6. *R_i is an affinoid k-algebra.*

Proof. The étale map $U_i \to \mathbb{T}^d$ induces an étale morphism $k\left\langle T_1^{\pm}, \ldots, T_d^{\pm}\right\rangle \to R_i$ of Huber rings. Now apply [37, Proposition 1.7.1(iii) and Corollary 1.7.2(ii)]. □

Lemma 3.6.7. *Let A denote a k_0-affinoid algebra and fix a finite set of elements $s_1, \ldots, s_n \in A$. Set $I := (g_1, \ldots, g_n)$ and consider*

$$\omega^q \colon A\left\langle \frac{s_1, \ldots, s_n}{p^q}\right\rangle \to \varprojlim_l A/I^l$$

for all $q \in \mathbb{N}$. Then the $\ker\omega^q$ are Banach spaces and $\varinjlim_q \ker\omega^q = 0$ as a k_0-ind-Banach space.

Proof. Write $I_q := A\left\langle \frac{s_1,\ldots,s_n}{\pi^q}\right\rangle I$. Then

$$A\left\langle \frac{s_1, \ldots, s_n}{\pi^q}\right\rangle / I_q^l \xrightarrow{\cong} A/I^l$$

for all l, thus the kernel of ω^q is the intersection of the ideals I_q^l. But any ideal in an affinoid algebra is closed, see for example [22, section 6.1.1 Proposition 3]. Therefore $\ker\omega^q$ is closed, thus complete. In particular, we may view it as a Banach space with the restriction of the norm on $A\left\langle \frac{s_1,\ldots,s_n}{\pi^q}\right\rangle$.

By the Krull intersection theorem, there exists an element $f \in I_q$ such that

$$(1-f)\ker\omega_m = (1-f)\bigcap_l I_q^l = 0.$$

Now consider the commutative diagram

$$\begin{array}{ccc} A\left\langle \frac{\zeta_1,\ldots,\zeta_n}{\pi^q}\right\rangle & \xrightarrow{\upsilon^q} & A\left\langle \frac{s_1,\ldots,s_n}{\pi^q}\right\rangle \\ \downarrow{\scriptstyle \widetilde{\iota}_{q,q'}} & & \downarrow{\scriptstyle \iota_{q,q'}} \\ A\left\langle \frac{\zeta_1,\ldots,\zeta_n}{\pi^{q'}}\right\rangle & \xrightarrow{\upsilon^{q'}} & A\left\langle \frac{s_1,\ldots,s_n}{\pi^{q'}}\right\rangle \end{array}$$

for any $q' \geq q$, where the horizontal maps send each ζ_i to s_i. Pick $\widetilde{f}$ such that $\upsilon^q\left(\widetilde{f}\right) = f$. Since $f \in I_q$, we may assume that $\widetilde{f} \in (\zeta_1, \ldots, \zeta_n)$. In particular,

$$\|\iota_{q,q'}(f)\| \leq \|\tau^{q'}\left(\widetilde{\iota}_{q,q'}\left(\widetilde{f}\right)\right)\| \leq \|\widetilde{\iota}_{q,q'}\left(\widetilde{f}\right)\| \leq |\pi|^{q'-q}\|\widetilde{f}\|.$$

That is, we may chose q' large enough such that $\|\iota_{q,q'}(f)\| \leq 1$. But then $\iota_{q,q'}(1-f)$ becomes a unit: its inverse is $\sum_{\alpha\geq 0}\iota_{q,q'}(f)^\alpha$. Thus $\iota_{q,q'}(\ker\omega^q)$ is killed by a unit, thus it is zero. In other words, $\ker\omega^q \subseteq \ker\iota^{q,q'}$. It follows that

$$\varinjlim_{q\geq 0}\ker\omega^q \subseteq \varinjlim_{q'\geq q\geq 0}\ker\iota_{q,q'} = \ker\varinjlim_{q'\geq q\geq 0}\iota_{q,q'} = 0.$$

But $\iota := \varinjlim_{m' \geq m \geq 0} \iota_{q,q'}$ is by definition an automorphism of $\varinjlim_{m \geq 0} A \left\langle \frac{s_1, \ldots, s_n}{\pi^q} \right\rangle$, thus $\ker \iota = 0$. This implies $\varinjlim_{q \geq 0} \ker \omega^q = 0$, as desired. □

Notation 3.6.8. Denote the multiplication $R_i \widehat{\otimes}_{k_0} R_i \to R_i$ by μ_i.

Proposition 3.6.9. *Fix generators $(s_1, \ldots, s_n) = \ker \mu_i$. We have an isomorphism*

$$\tau_i \colon R_i \left\langle \frac{Z_1, \ldots, Z_d}{p^\infty} \right\rangle \xrightarrow{\cong} \left(R_i \widehat{\otimes}_{k_0} R_i\right) \left\langle \frac{s_1, \ldots, s_n}{p^\infty} \right\rangle$$

extending $r \mapsto 1 \widehat{\otimes} r$ and sending Z_l to $T_l \widehat{\otimes} 1 - 1 \widehat{\otimes} T_l$ of k-ind-Banach algebras.

Proof. By [42, Lemma 2.3] the canonical morphism

$$\left(R_i \widehat{\otimes}_{k_0} R_i\right) \left\langle \frac{s_1, \ldots, s_n}{p^\infty} \right\rangle \xrightarrow{\cong} \left(R_i \widehat{\otimes}_{k_0} R_i\right)_U$$

is an isomorphism of k-ind-Banach algebras. Here, the colimit on the left runs through the system of affinoid open neighbourhoods $U \supseteq \Delta\left(\operatorname{Sp}\left(R_i\right)\right)$ and $(-)_U$ denotes the corresponding localisation. In particular, the constructions in the [58, proof of Proposition 6.31] apply. *Loc. cit.* constructs a morphism [3]

$$\sigma_i \colon \left(R_i \widehat{\otimes}_{k_0} R_i\right) \left\langle \frac{s_1, \ldots, s_n}{p^\infty} \right\rangle \longrightarrow R_i \left\langle \frac{Z_1, \ldots, Z_d}{p^\infty} \right\rangle,$$

such that $\sigma_i \circ \tau_i$ is the identity. [52, Proposition 1.1.8] implies that σ_i is a strict epimorphism, thus it remains to check that it is a monomorphism. To do this, we apply Lemma 3.6.7 for $A := R_i \widehat{\otimes}_{k_0} R_i$ and $I := \ker \mu_i$. *Loc. cit.* considers the morphism ω_i, which fit into the following commutative diagrams:

$$\begin{array}{ccc} \left(R_i \widehat{\otimes}_{k_0} R_i\right) \left\langle \frac{s_1, \ldots, s_n}{p^q} \right\rangle & \xrightarrow{\tau_i^q} & R_i \left\langle \frac{Z_1, \ldots, Z_d}{p^\infty} \right\rangle \\ \downarrow \omega_i^q & & \downarrow \\ \varprojlim_j \left(R_i \widehat{\otimes}_{k_0} R_i\right) / \left(\ker \mu_i\right)^j & \xrightarrow{\cong} & R_i \left[\!\left[Z_1, \ldots, Z_d\right]\!\right]. \end{array} \tag{3.6.8}$$

The morphisms τ_i^q at the top are the compositions

$$\left(R_i \widehat{\otimes}_{k_0} R_i\right) \left\langle \frac{s_1, \ldots, s_n}{p^q} \right\rangle \longrightarrow \left(R_i \widehat{\otimes}_{k_0} R_i\right) \left\langle \frac{s_1, \ldots, s_n}{p^\infty} \right\rangle \xrightarrow{\sigma_i} R_i \left\langle \frac{Z_1, \ldots, Z_d}{p^\infty} \right\rangle,$$

and the isomorphism at the bottom of the commutative diagram comes from [58, Lemma B.7 and Corollary B.10]. The commutative diagram (3.6.8) implies $\ker \tau_i^q = \ker \omega_i^q$. Together with the aforementioned Lemma 3.6.7, we find

$$\ker \tau_i = \varinjlim_q \ker \tau_i = \varinjlim_q \ker \omega_i^q = 0,$$

as desired. □

[3] [58] denotes σ_i by ψ and τ_i by φ.

Proof of Theorem 3.5.5. As explained at the beginning of this subsection 3.6, it suffices to check that the morphism (3.6.1) is an isomorphism.

Recall that $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(R,R^+)$ is an R_i-Banach algebra via ϵ_i, cf. Notation 3.6.3. See Proposition 3.6.9 for τ_i and write $\psi_i := \mathrm{id}_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(R,R^+)} \widehat{\otimes}_{R_i} \tau_i$. We claim

$$\phi_i \cong \psi_i.$$

Because ψ_i is an isomorphism, this would imply that ϕ_i is an isomorphism.

Step 1. The domain of ψ_i coincides with the domain of ϕ_i:

$$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(R,R^+\right)\widehat{\otimes}_{R_i}R_i\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle \overset{2.4.7}{\cong} \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(R,R^+\right)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle.$$

Step 2. We compute the codomain of ψ_i. Recall the generating set G^+ of $\ker\mathcal{O}\theta_{\mathrm{inf}}$ considered in Lemma 3.6.5. Additionally, fix a generating set S of $\ker\mu_i^+$. By [2, Theorem 4.1.4], we may assume that $S=\{s_1,\ldots,s_n\}$ is finite. Define

$$G := \left(\mathrm{id}_{R_i}\widehat{\otimes}_{k_0}\epsilon_i\right)(s_1,\ldots,s_n)\cup\left\{1\widehat{\otimes}\xi\right\}\subseteq R_i\widehat{\otimes}_{k_0}\widehat{\mathbb{B}}_{\mathrm{inf}}\left(R,R^+\right)$$

and compute, for every $q\in\mathbb{N}$,

$$\begin{aligned}
&\mathbb{B}_{\mathrm{dR}}^{q,+}\left(R,R^+\right)\widehat{\otimes}_{R_i}\left(R_i\widehat{\otimes}_{k_0}R_i\right)\left\langle\frac{S}{p^q}\right\rangle \\
&\overset{3.3.24}{\cong} \operatorname{coker}\left(\widehat{\mathbb{B}}_{\mathrm{inf}}\left(R,R^+\right)\left\langle\frac{\zeta}{p^q}\right\rangle\left(p^q\frac{\zeta}{p^q}-\xi\right)\to\widehat{\mathbb{B}}_{\mathrm{inf}}\left(R,R^+\right)\left\langle\frac{\zeta}{p^q}\right\rangle\right) \\
&\qquad\widehat{\otimes}_{R_i}\operatorname{coker}\left(\bigoplus_{j=1}^n\left(R_i\widehat{\otimes}_{k_0}R_i\right)\left\langle\frac{\zeta_s}{p^q}\colon s\in S\right\rangle\left(p^q\frac{\zeta_{s_j}}{p^q}-s_j\right)\to\left(R_i\widehat{\otimes}_{k_0}R_i\right)\left\langle\frac{\zeta_s}{p^q}\colon s\in S\right\rangle\right) \\
&\cong\operatorname{coker}\left(\bigoplus_{g\in G}\left(R_i\widehat{\otimes}_{k_0}\widehat{\mathbb{B}}_{\mathrm{inf}}\left(R,R^+\right)\right)\left\langle\frac{\zeta_g}{p^q}\colon g\in G\right\rangle\left(p^q\frac{\zeta_g}{p^q}-g\right)\to\left(R_i\widehat{\otimes}_{k_0}\widehat{\mathbb{B}}_{\mathrm{inf}}\left(R,R^+\right)\right)\left\langle\frac{\zeta_g}{p^q}\colon g\in G\right\rangle\right) \\
&\cong\operatorname{coker}\left(\bigoplus_{g\in G^+}\left(R_i^+\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}\left(R,R^+\right)\right)\left\langle\frac{\zeta_f}{p^q}\colon g\in G^+\right\rangle\left(p^q\frac{\zeta_g}{p^q}-g\right)\to\left(R_i^+\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}\left(R,R^+\right)\right)\left\langle\frac{\zeta_g}{p^q}\colon g\in G^+\right\rangle\right)\widehat{\otimes}_{W(\kappa)}k_0 \\
&=\left(R_i^+\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}\left(R,R^+\right)\right)\left\langle\frac{G^+}{p^q}\right\rangle\widehat{\otimes}_{W(\kappa)}k_0 \\
&\overset{2.5.10}{\cong}\left(R_i^+\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}\left(R,R^+\right)\right)\left\langle\frac{\ker\mathcal{O}\theta_{\mathrm{inf}}}{p^q}\right\rangle\widehat{\otimes}_{W(\kappa)}k_0.
\end{aligned}$$

Now pass to the colimit along $q \to \infty$ to see that ϕ_i and ψ_i have the same codomain:

$$
\begin{aligned}
&\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(R,R^+\right)\widehat{\otimes}_{R_i}\left(R_i\widehat{\otimes}_{k_0}R_i\right)\left\langle\frac{S}{p^\infty}\right\rangle \\
&\cong\left(R_i^+\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}\left(R,R^+\right)\right)\left\langle\frac{\ker\mathcal{O}\theta_{\mathrm{inf}}}{p^\infty}\right\rangle\widehat{\otimes}_{W(\kappa)}k_0.
\end{aligned}
$$

Step 3. Both ϕ_i and ψ_i are colimits of completed localisations of morphisms

$$
\mathbb{A}_{\mathrm{dR}}^q\left(R,R^+\right)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle\to\left(R_i^+\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U)\right)\left\langle\frac{\ker\mathcal{O}\theta_{\mathrm{inf}}}{p^q}\right\rangle
$$

of $\mathbb{A}_{\mathrm{dR}}^q\left(R,R^+\right)$-Banach algebras, $q\in\mathbb{N}$. Therefore, it suffices to check that both ϕ_i and ψ_i coincide on the variables $Z_1,\ldots,Z_d$. This is an easy computation:

$$
\begin{aligned}
\phi_i\left(Z_l\right)&=T_l\widehat{\otimes}_{W(\kappa)}1-1\widehat{\otimes}_{W(\kappa)}\left[T_l^\flat\right],\ \text{and}\\
\psi_i\left(Z_l\right)&=\left(\mathrm{id}_{R_i^+}\widehat{\otimes}_{W(\kappa)}\epsilon_i^+\right)\left(\tau_i\left(Z_l\right)\right)\\
&=\left(\mathrm{id}_{R_i^+}\widehat{\otimes}_{W(\kappa)}\epsilon_i^+\right)\left(T_l\widehat{\otimes}_{W(\kappa)}1-1\widehat{\otimes}_{W(\kappa)}T_l\right)\\
&=T_l\widehat{\otimes}_{W(\kappa)}1-1\widehat{\otimes}_{W(\kappa)}\left[T_l^\flat\right]
\end{aligned}
$$

for all $l=1,\ldots,d$. □

Chapter 4

Differential operators meet p-adic Hodge theory

Fix the notation introduced in subsection 3.1.

4.1 Infinite order differential operators on rigid-analytic spaces

X denotes a smooth rigid-analytic k-variety. We recall the construction of the sheaf $\widehat{\mathcal{D}}$ on X. Our main reference is [7]. X_w denotes the category whose objects are the affinoid subdomains of X and whose morphisms are the inclusions, carrying the weak Grothendieck topology. The *d-dimensional torus over k* is

$$\mathbb{T}^d := \operatorname{Sp}\left(k\left\langle T_1^{\pm}, \ldots, T_d^{\pm}\right\rangle\right).$$

First, we assume that $X = \operatorname{Sp} A$ is affinoid and equipped with an étale morphism $g\colon X \to \mathbb{T}^d$. Compute the A-module of k-linear differentials of A:

$$L := \operatorname{Der}_k(A) = \bigoplus_{l=1}^{d} A\partial_l,$$

where ∂_l denotes the lift of the canonical vector field d/dT_l along the étale map $g^{\#}\colon \mathcal{O}\left(\mathbb{T}^d\right) = k\left\langle T_1^{\pm}, \ldots, T_d^{\pm}\right\rangle \to \mathcal{O}(X)$. These ∂_l do not need to preserve $\mathcal{A} := \mathcal{O}(X)^{\circ}$ in general. But because they are bounded, see [4, Lemma 3.1], we can find $r_g \geq 1$ large enough such that the $\pi^r\partial_l$ preserve $\mathcal{A}$ for every $r \geq r_g$. Define the $\mathcal{A}$-submodule

$$\mathcal{L} := \bigoplus_{l=1}^{d} \mathcal{A}\partial_l \subseteq L.$$

$\mathcal{L}_r := p^r\mathcal{L}$ is an $\mathcal{A}$-*Lie lattice* for every $r \geq r_g$, that is $[\mathcal{L}_r, \mathcal{L}_r] \subseteq \mathcal{L}_r$ and $\mathcal{L}_r(\mathcal{A}) \subseteq \mathcal{A}$.

Remark 4.1.1. A priori, it seems more natural to define $\mathcal{L}_r$ to be $\pi^r\mathcal{L}$. However, we choose p over π because it simplifies the proof of Theorem 4.2.1.

We cite the following from [7, section 3]. Fix $r \geq r_g$. Denote the pullback $\mathcal{O}(X) \to \mathcal{O}(U)$ by ω. An *admissible* k°-algebra is a commutative k°-algebra which is topologically of finite type and flat over k°. An *affine formal model* in $\mathcal{O}(U)$ is an admissible k°-algebra $\mathcal{B}$ such that $\mathcal{O}(U) \cong \mathcal{B} \otimes_{k^\circ} k$. $\mathcal{B}$ is $\mathcal{L}_r$*-stable* if $\omega(\mathcal{A}) \subseteq \mathcal{B}$ and the action of $\mathcal{L}_r$ on $\mathcal{A}$ lifts to $\mathcal{B}$. U is $\mathcal{L}_r$*-admissible* if it admits an $\mathcal{L}_r$-stable affine formal model. $X_r = X_{g,r} := X_w(\mathcal{L}_r)$ denotes the full subcategory of X_w consisting of the $\mathcal{L}_r$-admissible affinoid subdomains. It is a site by [7, Subsection 3.2, Lemma]. The coverings are the finite admissible coverings by objects in X_r.

Definition 4.1.2. [7, Section 3.3, Definition] Let X be affinoid and equipped with an étale morphism $g\colon X \to \mathbb{T}^d$. Fix $r \geq r_g$. For any $\mathcal{L}_r$-admissible affinoid subdomain $U \subseteq X$ and any $\mathcal{L}_r$-stable affine formal model $\mathcal{B}$ in $\mathcal{O}(U)$, define

$$\mathcal{D}_r(U) = \mathcal{D}_{g,r}(U) := \widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_r)} \otimes_{k^\circ} k.$$

The symbol U refers to the enveloping algebra of $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_r$ as a $(k^\circ, \mathcal{B})$-algebra, see [7, subsection 2.1]. The completion is the π-adic one.

Regard $\mathcal{D}_r(U)$ as a k-Banach algebra with unit ball $\widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_t)}$.

Lemma 4.1.3. *Let X be affinoid and equipped with an étale morphism $g\colon X \to \mathbb{T}^d$. Fix $r \geq r_g$. The assignment*

$$U \mapsto \mathcal{D}_r(U)$$

defines a sheaf of k-Banach algebras on X_r.

Proof. See [7, section 3.5, Theorem] and apply the open mapping theorem. □

Definition 4.1.4. [7, Section 9.3, Definition] X is affinoid and it admits an étale morphism $X \to \mathbb{T}^d$. For every $U \in X_w$, define the k-ind-Banach algebra

$$\widehat{\mathcal{D}}(U) := \varprojlim \mathcal{D}_{g,r}(U),$$

cf. Lemma A.0.5. The inverse limits runs over all étale maps $g\colon X \to \mathbb{T}^d$ and r large enough such that $U \in X_r$.

Lemma 4.1.5. *X is affinoid and admits an étale morphism $X \to \mathbb{T}^d$. Then*

$$0 \to \widehat{\mathcal{D}}(X) \to \prod_j \widehat{\mathcal{D}}(U_j) \to \prod_{j_1,j_2} \widehat{\mathcal{D}}(U_{j_1} \times_U U_{j_2})$$

is strictly exact for any finite covering $\{U_j \to X\}_j$ by affinoids.

Proof. This follows from Lemma 4.1.3. □

Definition 4.1.6. Let X be an arbitrary smooth rigid-analytic k-variety. By abuse of notation, $\widehat{\mathcal{D}}$ denotes the unique sheaf of k-ind-Banach algebras such that for every $U \in X_w$ admitting an étale map $U \to \mathbb{T}^d$, $\widehat{\mathcal{D}}(U)$ is given as in Definition 4.1.4.

Lemma 4.1.7. *Suppose X is affinoid and equipped with a fixed étale morphism $g\colon X \to \mathbb{T}^d$. Then the canonical morphism $\widehat{\mathcal{D}}(U) \xrightarrow{\cong} \varprojlim_{r \geq r_g} \mathcal{D}_{g,r}(U)$ is an isomorphism for any affinoid subdomain $U \subseteq X$.*

Proof. This follows from [7, section 6.1, Lemma, (b)]. □

See [7, section 8 and 9] for the definition of the category of *coadmissible $\widehat{\mathcal{D}}$-modules* as a full subcategory of abstract $\widehat{\mathcal{D}}$-modules. [18] realised it as a full subcategory of the category of sheaves of complete bornological $\widehat{\mathcal{D}}$-modules, and thus as a full subcategory of the category of sheaves of $\widehat{\mathcal{D}}$-ind-Banach modules. We are interested in a derived analog. Recall that every quasi-abelian category has a derived category, cf. [52, section 1.2]. See also *loc. cit.* section 1.3 on derivation of functors.

Definition 4.1.8. Consider a closed monoidal quasi-abelian category $\mathbf{E}$, admitting a ring object R. Then the category of R-modules is again quasi-abelian, cf. [52, Proposition 1.5.1.], thus it admits a derived category $\mathbf{D}(R)$. The *category $\mathbf{D}^{\mathrm{b}}_{\mathrm{perf}}(R)$ of bounded perfect complexes of R-modules* is the smallest full triangulated subcategory of $\mathbf{D}(R)$ which contains R, and is closed under direct summands and isomorphisms. A *bounded perfect complex of R-modules* is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{perf}}(R)$.

Definition 4.1.9. [18, Section 6] An object $\mathcal{M}^\bullet$ in the derived category of sheaves of $\widehat{\mathcal{D}}$-ind-Banach modules is a *$\mathcal{C}$-complex* if there exists an admissible covering of X by affinoids X_i equipped with étale morphisms $g_i\colon X_i \to \mathbb{T}^{d_i}$ such that, for each i,

(i) $\mathcal{M}^\bullet_r := \mathcal{D}_{g_i,r} \widehat{\otimes}^{\mathrm{L}}_{\widehat{\mathcal{D}}|_{X_{i,r}}} \mathcal{M}^\bullet|_{X_{i,r}}$ is a bounded perfect complex of sheaves of $\mathcal{D}_{g_i,r}$-ind-Banach modules for every $r \geq r_{g_i}$, and

(ii) $\mathrm{H}^j(\mathcal{M}^\bullet) \xrightarrow{\cong} \varprojlim_{r \geq r_{g_i}} \mathrm{H}^j(\mathcal{M}^\bullet_r)$ for every $j \in \mathbb{Z}$.

$\mathbf{D}_{\mathcal{C}}\left(\widehat{\mathcal{D}}\right) \subseteq \mathbf{D}_{\mathcal{C}}\left(\widehat{\mathcal{D}}\right)$ denotes the full triangulated subcategory of $\mathcal{C}$-complexes.

Remark 4.1.10. We remind the reader that the cohomology of a complex $\mathcal{N}^\bullet \in \mathbf{D}\left(\widehat{\mathcal{D}}\right)$ is not an object of the category $\mathbf{Mod}\left(\widehat{\mathcal{D}}\right)$ but its *left heart*, cf. [52, section 1.2.2].

Remark 4.1.11. [18, Section 6] requires each $\mathcal{M}^\bullet_r$ to be a bounded complex of coherent $\mathcal{D}_{g_i,r}$-modules, but we require it to be a bounded perfect complex. Both definitions are equivalent by the main result of [20].

4.2 The bimodule structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger}$

We continue to fix a smooth rigid-analytic k-variety X. Every affinoid k-algebra is strongly Noetherian, cf. [57, Theorem 3.1.8.3] and [22, section 6.1.1, Proposition 3], thus the adic space X^{ad} associated to X is locally Noetherian. This allows to define $X_{\mathrm{pro\acute{e}t}} := \left(X^{\mathrm{ad}}\right)_{\mathrm{pro\acute{e}t}}$, the *pro-étale site of* X. The morphism of sites ν is

$$X_{\mathrm{pro\acute{e}t}} = \left(X^{\mathrm{ad}}\right)_{\mathrm{pro\acute{e}t}} \stackrel{\nu}{\longrightarrow} \left(X^{\mathrm{ad}}\right)_{\mathrm{\acute{e}t}} \longrightarrow X.$$

See [37, section 2.1] for the definition of the morphism at the right-hand side.

The term *module* always refers to a left module. We view $\nu^{-1}\widehat{\mathcal{D}}$ as a sheaf of k-ind-Banach algebras and $\nu^{-1}\mathcal{O}$ as a sheaf of $\nu^{-1}\widehat{\mathcal{D}}$-ind-Banach modules, cf. Lemma B.1.6. This makes $\nu^{-1}\mathcal{O}\widehat{\otimes}_{k_0}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ a $\nu^{-1}\widehat{\mathcal{D}}\widehat{\otimes}_{k_0}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-module object. Since $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ is commutative, it is in fact a $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-bimodule object, cf. Definition A.0.3.

Theorem 4.2.1. *There exists a* $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$*-bimodule structure on* $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ *such that the canonical morphism*

$$\nu^{-1}\mathcal{O}\widehat{\otimes}_{k_0}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \tag{4.2.1}$$

is a morphism of $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$*-bimodule objects. It is unique.*

Remark 4.2.2. Here is an overview of the proof of Theorem 4.2.1. We give a local construction of the bimodule action. Assume that X is affinoid and equipped with an étale morphism $X \to \mathbb{T}^d$. We have the elements $\partial_1, \ldots, \partial_d \in \widehat{\mathcal{D}}(X)$. Cf. Theorem 3.5.5, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ has the sections $Z_1, \ldots, Z_d$, locally on the proétale site.

$$\partial_j \cdot Z_l := \frac{d}{dZ_j}(Z_l) = \delta_{jl}$$

defines the action of $\nu^{-1}\widehat{\mathcal{D}}$, where δ_{jl} is the Kronecker delta. A large part of the proof concerns showing that this action is compatible with the canonical $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-module structure. Finally, we use that (4.2.1) is an epimorphism to show uniqueness.

Proof of Theorem 4.2.1. Corollary A.0.4 and the Lemma 4.2.3 imply uniqueness.

Lemma 4.2.3. *The morphism (4.2.1) is an epimorphism.*

Proof. For every $q \in \mathbb{N}$, $U = \text{“}\varprojlim_{i\in I}\text{”}U_i \in X_{\mathrm{pro\acute{e}t}}$, and $i \in I$,

$$\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{dR}}^q(U) \to \left(\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U)\right)\left\langle\frac{\ker \mathcal{O}\theta_{\mathrm{inf}}}{p^q}\right\rangle$$

has dense image. It is an epimorphism by Lemma 2.1.7(ii).

$$\text{“}\varinjlim_{i\in I}\text{”}\mathcal{O}(U_i)\widehat{\otimes}_{k_0}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\to\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$$

is an epimorphism because colimits preserve epimorphisms. Now sheafify and apply Lemma B.1.3(iii) and B.1.5, to find that (4.2.1) is an epimorphism. □

It remains to construct the bimodule structure. We may work locally, as the uniqueness of the bimodule structures ensures they will glue. Therefore, we can assume that X is affinoid and equipped with an étale morphism $g\colon X\to\mathbb{T}^d$.

$\widetilde{X}\to X$ is the pro-étale covering as in Theorem 3.5.5. Let $U\in X_{\text{proét}}/\widetilde{X}$. One can assume that U is affinoid perfectoid by Lemma 3.2.6. We aim to give $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$ the structure of a $\nu^{-1}\widehat{\mathcal{D}}(U)$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$-bimodule object, functorially in U. This is equivalent to giving $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$ the structure of a $\widehat{\mathcal{D}}(V)$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$-bimodule object for every $V\in X_w$ with $U\to\nu^{-1}(V)$, functorial in U and V. By Corollary 3.5.7,

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\cong\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^\infty}\right\rangle\overset{2.4.7}{\cong}\varinjlim_{q\in\mathbb{N}}\mathbb{A}_{\mathrm{dR}}^{q}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle\widehat{\otimes}_{W(\kappa)}k_0.$$

Apply Lemma 4.1.7 to get

$$\begin{aligned}\widehat{\mathcal{D}}(V)&\cong\varprojlim_{r\geq r_g}\mathcal{D}_{g,r}(V)=\varprojlim_{r\geq r_g}\widehat{U(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}_r)}\otimes_{k^\circ}k\\&\overset{2.3.2}{\cong}\varprojlim_{r\geq r_g}\widehat{U(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}_r)}[1/p]\overset{2.3.6}{\cong}\varprojlim_{r\geq r_g}\widehat{U(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}_r)}\widehat{\otimes}_{W(\kappa)}k_0.\end{aligned}$$

$\mathcal{A}$ is an affine formal model in $\mathcal{O}(X)$, $\mathcal{L}:=\bigoplus_{l=1}^{d}\mathcal{A}\partial_l$ where ∂_l denotes the lift of the canonical vector field d/dT_l along the étale map $g^{\#}\colon\mathcal{O}(\mathbb{T}^d)=k\langle T_1^{\pm},\ldots,T_d^{\pm}\rangle\to\mathcal{O}(X)$, $\mathcal{L}_r:=p^r\mathcal{L}$, and $\mathcal{B}$ is an $\mathcal{L}_r$-stable affine formal model in $\mathcal{O}(V)$. Assume that r is sufficiently large, such that $V\subseteq X$ is $\mathcal{L}_r$-admissible. This is possible by [7, section 6.1, Lemma (b)]. We construct $\widehat{U(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}_r)}$-$\mathbb{A}_{\mathrm{dR}}^{q}(U)$-bimodule structures on $\mathbb{A}_{\mathrm{dR}}^{q}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle$ for $r,q\in\mathbb{N}$ large enough with $q\leq r$, such that the following holds.

Condition 4.2.4.

(i) For all $r'\geq r$, the following diagrams commute:

$$\begin{array}{ccc}\left(\widehat{U(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}_r)}\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{dR}}^{q}(U)\right)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{dR}}^{q}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle & \longrightarrow & \mathbb{A}_{\mathrm{dR}}^{q}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle\\ \uparrow & & \|\\ \left(\widehat{U(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}_{r'})}\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{dR}}^{q}(U)\right)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{dR}}^{q}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle & \longrightarrow & \mathbb{A}_{\mathrm{dR}}^{q}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle.\end{array}$$

Here, the horizontal maps denote the bimodule structures.

(ii) For all $q' \geq q$, the following diagrams commute:

$$\begin{array}{ccc} W(\kappa) & \longrightarrow & \mathbb{A}_{\mathrm{dR}}^{q'}(U)\left\langle \frac{Z_1,\ldots,Z_d}{p^{q'}} \right\rangle \\ \Vert & & \uparrow \\ W(\kappa) & \longrightarrow & \mathbb{A}_{\mathrm{dR}}^{q}(U)\left\langle \frac{Z_1,\ldots,Z_d}{p^{q}} \right\rangle \end{array}$$

Here, the horizontal maps denote the unit maps.

Once these bimodule structures are constructed, invert p to get a $\mathcal{D}_{g,r}(V)-\mathbb{B}_{\mathrm{dR}}^{q,+}(U)$-bimodule structure on $\mathbb{B}_{\mathrm{dR}}^{q,+}(U)\left\langle \frac{Z_1,\ldots,Z_d}{p^q} \right\rangle$. Lemma A.0.7 applies because of Condition 4.2.4, giving the desired $\widehat{\mathcal{D}}(V)$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$-bimodule structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)$.

Simplify notation: $O := \mathcal{O}(V)^\circ$, $L_r := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_r$, $A^q := \mathbb{A}_{\mathrm{dR}}^q(U)$, and $OA^q := \mathbb{A}_{\mathrm{dR}}^q(U)\left\langle \frac{Z_1,\ldots,Z_d}{p^q} \right\rangle$. We explained above that we aim to construct a $\widehat{U(L_r)}$-A^q-bimodule structure on OA^q. This is equivalent to giving a morphism

$$\widehat{U(L_r)} \to \underline{\mathrm{Hom}}_{A^q}(OA^q, OA^q) \tag{4.2.2}$$

of $W(\kappa)$-Banach algebras. We also explained that the morphism has to be suitably functorial in V and U. This will be obvious from the construction. Thus we choose to omit both U and V in the simplified notation.

View L_r together with the canonical anchor map $\sigma\colon L_r \to \mathrm{Der}_{k^\circ}(O)$ as a (k°, O)-Lie-algebra, cf. [7, section 2.1]. We construct (4.2.2) via the universal property of the completed enveloping algebra. Without loss of generality, $q \in \mathbb{N}_{\geq 1}$. Write

$$\begin{aligned} j_O\colon O &\to \underline{\mathrm{Hom}}_{A^q}(OA^q, OA^q), \\ f &\mapsto (h \mapsto \widetilde{\epsilon}(f)\, h), \end{aligned}$$

where $\widetilde{\epsilon}$ denotes the map constructed in Lemma 3.6.1, but we omit the index i. Indeed, the image of $\widetilde{\epsilon}$ lies in $\mathbb{A}_{\mathrm{inf}}(R, R^+)[\![Z_1, \ldots, Z_d]\!] \subseteq OA^q$ because $q \geq 1$. Next,

$$\begin{aligned} j_{L_r}\colon L_r &\to \underline{\mathrm{Hom}}_{A^q}(OA^q, OA^q), \\ \sum_{l=1}^d f_l p^r \partial_l &\mapsto \sum_{l=1}^d \widetilde{\epsilon}(f_l)\, p^r \frac{d}{dZ_l} \end{aligned}$$

is well-defined because $q \leq r$. Indeed, compute for every $l = 1, \ldots, d$ and $\alpha \in \mathbb{N}^d$,

$$\begin{aligned} j_{L_r}(p^r \partial_l)\left(\left(\frac{Z}{p^q}\right)^\alpha\right) &= p^r \frac{d}{dZ_l}\left(\left(\frac{Z}{p^q}\right)^\alpha\right) \\ &= \frac{p^r}{p^q}\alpha_l \left(\frac{Z}{p^{(q-1)}}\right)^{\alpha_l - 1}\left(\frac{Z}{p^q}\right)^{\alpha - \alpha_l e_l} \in OA^q, \end{aligned}$$

where $e_l = (0, \ldots, 1, \ldots, 0)$ denotes the lth unit vector. Then one shows directly that $j_{L_r}(p^r \partial_l)$ defines a bounded linear map $OA^q \to OA^q$. To exploit the universal property of $U(L_r)$, the following has to be checked:

Condition 4.2.5.

(a) j_O is a homomorphism of k°-algebras.

(b) j_{L_r} is an O-Lie algebra homomorphism.

(c) For all $f \in O$ and $P \in L_r$, $j_{L_r}(fP) = j_O(f)\, j_{L_r}(P)$.

(d) For all $f \in O$ and $P \in L_r$, $[j_{L_r}(P), j_O(f)] = j_O(\sigma(P)(f))$.

By [7, section 2.1], this would indeed give a continuous map

$$U(L_r) \to \underline{\mathrm{Hom}}_{A^q}(OA^q, OA^q)$$

of $W(\kappa)$-algebras. It would extend by continuity to the desired morphism (4.2.2).

Conditions 4.2.5 (a) and (c) are obvious. It remains to check (b) and (c).

Lemma 4.2.6. *A bounded $W(\kappa)$-linear derivation $D\colon O \to OA^q$ is a bounded $W(\kappa)$-linear map which satisfies the Leibniz rule. If $D|_{\mathcal{O}(\mathbb{T}^d)^\circ} = 0$, then $D = 0$.*

Proof. Equip both OA^q and $O = \mathcal{O}(V)^\circ$ with the p-adic norms. D is still continuous because it is $W(\kappa)$-linear. Recall Lemma 2.3.6. The following is a bounded derivation between two k-Banach spaces:

$$D[1/p]\colon \mathcal{O}(V) \to OA^q[1/p].$$

By [31, section 3.6, page 64], it is identified with a bounded linear map

$$\Omega_{\mathcal{O}(V)/k} \to OA^q[1/p]. \tag{4.2.3}$$

Loc. cit. identifies the composition of (4.2.3) and $\Omega_{\mathcal{O}(\mathbb{T}^d)/k} \cong \Omega_{\mathcal{O}(V)/k}$ with

$$D[1/p]|_{\mathcal{O}(\mathbb{T}^d)} = D|_{\mathcal{O}(\mathbb{T}^d)^\circ}[1/p] = 0.$$

Therefore (4.2.3) is the zero map and so is $D[1/p]$. $D = 0$ follows. □

Lemma 4.2.7. *The following diagram commutes:*

$$\begin{array}{ccc} & & \underline{\mathrm{Hom}}_{A^q}(OA^q, OA^q) \\ & \overset{j_{L_r}}{\nearrow} & \big\downarrow {\scriptstyle -\circ\widetilde{\epsilon}} \\ L_r & \xrightarrow[P \mapsto \widetilde{\epsilon} \circ \sigma(P)]{} & \underline{\mathrm{Hom}}_{W(\kappa)}(O, OA^q)\,. \end{array}$$

Proof of Lemma 4.2.7. Fix $P \in L_r$. Both $j_{L_r}(P) \circ \widetilde{\epsilon}$ and $\widetilde{\epsilon} \circ \sigma(P)$ are bounded $W(\kappa)$-linear derivations. Because of Lemma 4.2.6, it suffices to check

$$(\widetilde{\epsilon} \circ \sigma(P))\,|_{\mathcal{O}\left(\mathbb{T}_{k_0}\right)^\circ} = j_{L_r}(P)\,|_{\mathcal{O}\left(\mathbb{T}_{k_0}\right)^\circ}.$$

Fix the O-basis $p^r\partial_1, \dots, p^r\partial_d$ for L_r and write $P = \sum_{i=1}^d f_i p^r \partial_i$ with $f_i \in O$. Both $(\widetilde{\epsilon} \circ \sigma(-))\,|_{\mathcal{O}\left(\mathbb{T}_{k_0}\right)^\circ}$ and $j_{L_r}(-)\,|_{\mathcal{O}\left(\mathbb{T}_{k_0}\right)^\circ}$ are O-linear, thus it suffices to check

$$(\widetilde{\epsilon} \circ \sigma(p^r\partial_l))\,|_{\mathcal{O}\left(\mathbb{T}_{k_0}\right)^\circ} = j_L(p^r\partial_l)\,|_{\mathcal{O}\left(\mathbb{T}_{k_0}\right)^\circ}$$

for every $l = 1, \dots, d$. Fix such an l. Now computing

$$(\widetilde{\epsilon} \circ \sigma(p^r\partial_l))(T_j) = j_{L_t}(p^r\partial_l)(j_O(T_j)) \tag{4.2.4}$$

for every $j = 1, \dots, d$ suffices. This is a direct computation:

$$(\widetilde{\epsilon} \circ \sigma(p^r\partial_l))(T_j) = \widetilde{\epsilon}\left(p^r \frac{d}{dT_l}(T_j)\right) = p^r\delta_{lj},$$

where δ_{ij} denotes the Kronecker delta, and

$$j_{L_r}(p^r\partial_l)(\widetilde{\epsilon}(T_j)) = p^r \frac{d}{dZ_l}\left(\left[T_j^\flat\right] + Z_j\right) = p^r\delta_{lj}.$$

The last step uses that $\left[T_j^\flat\right] \in A^q$ is a constant and d/dZ_l is an A^q-linear derivation. This gives the identity 4.2.4, and thus finishes the proof of Lemma 4.2.7. □

We verify Condition 4.2.5(b) in the following.

Lemma 4.2.8. *The morphism j_{L_r} is an O-Lie algebra homomorphism.*

Proof. j_{L_r} is k°-linear. It remains to show that it preserves the Lie bracket:

$$[j_{L_r}(P), j_{L_t}(Q)] = j_{L_r}([P, Q]), \tag{4.2.5}$$

for any $P, Q \in L_r$. For all $j = 1, \dots, d$, it suffices to check

$$[j_{L_r}(P), j_{L_r}(Q)](Z_j) = j_{L_r}([P, Q])(Z_j)$$

because both sides are derivations. But then we would have to compute

$$[j_{L_r}(P), j_{L_r}(Q)] \circ \widetilde{\epsilon} = j_{L_r}([P, Q]) \circ \widetilde{\epsilon}. \tag{4.2.6}$$

Indeed, if (4.2.6) is true, we could compute

$$\begin{aligned}
\left[j_{L_r}(P), j_{L_r}(Q)\right](Z_j) &= \left[j_{L_r}(P), j_{L_r}(Q)\right]\left(\left[T_j^\flat\right] + Z_j\right) \\
&= \left(\left[j_{L_r}(P), j_{L_r}(Q)\right] \circ \widetilde{\epsilon}\right)(T_j) \\
&\overset{(4.2.6)}{=} \left(j_{L_r}([P,Q]) \circ \widetilde{\epsilon}\right)(T_j) \\
&= j_{L_r}([P,Q])\left(\left[T_j^\flat\right] + Z_j\right) \\
&= j_{L_r}([P,Q])(Z_j).
\end{aligned}$$

In particular, (4.2.6) implies (4.2.5). The following computation

$$\begin{aligned}
&\left[j_{L_r}(P), j_{L_r}(Q)\right] \circ \widetilde{\epsilon} \\
&= \left(j_{L_r}(P) \circ j_{L_r}(Q) - j_{L_r}(Q) \circ j_{L_r}(P)\right) \circ \widetilde{\epsilon} \\
&= j_{L_r}(P) \circ \left(j_{L_r}(Q) \circ \widetilde{\epsilon}\right) - j_{L_r}(Q) \circ \left(j_{L_r}(P) \circ \widetilde{\epsilon}\right) \\
&\overset{4.2.7}{=} \left(j_{L_r}(P) \circ \widetilde{\epsilon}\right) \circ \sigma(Q) - \left(j_{L_r}(Q) \circ \widetilde{\epsilon}\right) \circ \sigma(P) \\
&\overset{4.2.7}{=} \left(\widetilde{\epsilon} \circ \sigma(P)\right) \circ \sigma(Q) - \left(\widetilde{\epsilon} \circ \sigma(Q)\right) \circ \sigma(P) \\
&\overset{4.2.7}{=} \widetilde{\epsilon} \circ \left(\sigma(P) \circ \sigma(Q) - \sigma(Q) \circ \sigma(P)\right) \\
&= \widetilde{\epsilon} \circ \left(\sigma(P) \circ \sigma(Q) - \sigma(Q) \circ \sigma(P)\right) \\
&= \widetilde{\epsilon} \circ [\sigma(P), \sigma(Q)] \\
&= \widetilde{\epsilon} \circ \sigma([P,Q]) \\
&\overset{4.2.7}{=} j_{L_r}([P,Q]) \circ \widetilde{\epsilon}
\end{aligned}$$

checks (4.2.6). □

We verify Condition 4.2.5(d) in the following.

Lemma 4.2.9. *For all $f \in O$ and $P \in L_r$, $[j_{L_r}(P), j_O(f)] = j_O(\sigma(P)(f))$.*

Proof. Write $P = \sum_{l=1}^d f_l p^r \partial_l$. Compute for every $h \in OA^q$ and $l = 1, \ldots, d$,

$$\begin{aligned}
\left[j_O(f_l)\, p^r \frac{d}{dZ_l}, j_O(f)\right](h) &= j_O(f_l)\left[p^r \frac{d}{dZ_l}, j_O(f)\right](h) \\
&= \widetilde{\epsilon}(f_l)\left(p^r \frac{d}{dZ_l}\left(\widetilde{\epsilon}(f)\right)\right) h \\
&= \left(j_O(f_l)\, p^r \frac{d}{dZ_l}\right)\left(\widetilde{\epsilon}(f)\right) h.
\end{aligned}$$

This establishes the identity at the left-hand side here:

$$\left[j_{L_r}(P), j_O(f)\right](h) = \left(j_{L_r}(P) \circ \widetilde{\epsilon}\right)(f)h \overset{4.2.7}{=} \left(\widetilde{\epsilon} \circ \sigma(P)\right)(f)h = j_O(\sigma(P)(f))\, h.$$

The identity at the right-hand side comes from the definition of j_O. □

We have thus verified Condition 4.2.5. This gives the morphism (4.2.2) and thus the $\widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_r)}$-$\mathbb{A}^q_{\mathrm{dR}}(U)$-bimodule structure on $\mathbb{A}^q_{\mathrm{dR}}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle$. Invert p via $-\widehat{\otimes}_{W(\kappa)}k_0$. Lemma A.0.7 applies because Condition 4.2.4 is satisfied, giving the desired $\widehat{\mathcal{D}}(V)$-$\mathbb{B}^{\dagger,+}_{\mathrm{dR}}(U)$-bimodule structure on $\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}(U)$. It is functorial in both V and U, giving $\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}$ the structure of a $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}^{\dagger,+}_{\mathrm{dR}}$-bimodule object.

We discussed above that uniqueness is immediate once we have checked that the canonical morphism (4.2.1) is a morphism of sheaves of $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}^{\dagger,+}_{\mathrm{dR}}$-bimodule objects. Given V and U as above, we may check that

$$\mathcal{O}(V)\widehat{\otimes}_{k_0}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}(U) \to \mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}(U) \tag{4.2.7}$$

is a morphism of $\widehat{\mathcal{D}}(V)$-$\mathbb{B}^{\dagger,+}_{\mathrm{dR}}(U)$-bimodule objects. It is obtained by considering

$$\mathcal{B}\widehat{\otimes}_{W(\kappa)}\mathbb{A}^q_{\mathrm{dR}}(U) \to \mathbb{A}^q_{\mathrm{dR}}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle, \tag{4.2.8}$$

inverting p, and sheafification. By Lemma B.1.5, it suffices to check that (4.2.8) is a morphism of $\widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_r)}$-$\mathbb{A}^q_{\mathrm{dR}}(U)$-bimodule objects.

It is $\mathbb{A}^q_{\mathrm{dR}}(U)$-linear, and the actions of $\widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_r)}$ on both the domain and codomain of (4.2.8) are $\mathcal{B}$-linear. Thus it remains to compare the actions of the differentials $p^r\partial_1,\ldots,p^r\partial_d$. But we have étale coordinates on $\mathcal{O}(V)$, and it suffices to compare the actions on these. First, compute for all $T_j \in \mathcal{O}(V)$,

$$p^r\partial_l \cdot T_j = p^r\delta_{lj}.$$

Second, the map (4.2.8) sends T_j to $\left[T_j^\flat\right] + Z_j$. Cf. the proof of Lemma 4.2.7:

$$p^r\partial_l \cdot \left(\left[T_j^\flat\right] + Z_j\right) = p^r\delta_{lj}.$$

Thus (4.2.8) is a morphism of $\widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}_r)}$-$\mathbb{A}^q_{\mathrm{dR}}(U)$-bimodules. □

Corollary 4.2.10. *There exists a $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}^{\dagger}_{\mathrm{dR}}$-bimodule structure on $\mathcal{O}\mathbb{B}^{\dagger}_{\mathrm{dR}}$ such that the canonical morphism*

$$\nu^{-1}\mathcal{O}\widehat{\otimes}_{k_0}\mathbb{B}^{\dagger}_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}^{\dagger}_{\mathrm{dR}} \tag{4.2.9}$$

is a morphism of $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}^{\dagger}_{\mathrm{dR}}$-bimodule objects. It is unique.

Proof of Corollary 4.2.10. The bimodule structure on $\mathcal{O}\mathbb{B}^{\dagger}_{\mathrm{dR}}$ is obtained by taking the bimodule structure on $\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}$ as in Theorem 4.2.1 and inverting t, locally on the pro-étale site. Regarding uniqueness, note that the canonical morphism (4.2.9) is an epimorphism. This follows from Lemma 4.2.3 and because the completed tensor product preserves epimorphisms. Thus Corollary A.0.4 applies. □

4.3 A Poincaré Lemma

X still denotes a smooth rigid-analytic k-variety. We construct a $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-linear connection on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$, coming from its $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-bimodule structure.

Remark 4.3.1. Similarly, we may construct a $\mathbb{B}_{\mathrm{dR}}^{\dagger}$-linear connection on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger}$ coming from its $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger}$-bimodule structure. The discussion here in Subsection 4.3 goes through verbatim. This includes the formulation and proof of Theorem 4.3.8.

Definition 4.3.2. By [7, section 9.1, Proposition], there is a coherent sheaf $\mathcal{T}$ on X with $\mathcal{T}(U) = \mathrm{Der}_k\,\mathcal{O}(U)$ for every affinoid subdomain $U \subseteq X$: the *tangent sheaf*. Its sections are k-Banach spaces. In particular, the restrictionf of $\mathcal{T}$ to the site X_w of affinoid subdomains of X equipped with the weak Grothendieck topology is a sheaf of Banach spaces by Lemma 2.6.1 and the open mapping theorem. Apply Lemma 2.6.2 to view it as a sheaf of k-ind-Banach spaces on X_w. In fact, it is by construction a sheaf of $\mathcal{O}$-ind-Banach modules.

Definition 4.3.3. We define the following map $\nabla_{\mathrm{dR}}^{\dagger,+}\colon \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}}\nu^{-1}\Omega^1$ of sheaves of $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-ind-Banach modules. The $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-bimodule structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ from Theorem 4.2.1 gives a morphism

$$\nu^{-1}\widehat{\mathcal{D}} \to \underline{\mathcal{H}om}_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)$$

of sheaves of k_0-ind-Banach algebras. The proof of *loc. cit.* actually implies that this morphism is $\nu^{-1}\mathcal{O}$-linear. Thus it lifts to a morphism

$$\nu^{-1}\widehat{\mathcal{D}}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \underline{\mathcal{H}om}_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)$$

of sheaves of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$-ind-Banach algebras; here, the left-hand side becomes a monoid object through Lemma A.0.2. Compose it with the canonical map

$$\nu^{-1}\mathcal{T}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \nu^{-1}\widehat{\mathcal{D}}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}.$$

to obtain

$$\nu^{-1}\mathcal{T}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \underline{\mathcal{H}om}_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right).$$

Its dual appears in the following composition:

$$\begin{aligned}\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} &\to \underline{\mathcal{H}om}_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\underline{\mathcal{H}om}_{\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right), \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)\\ &\to \underline{\mathcal{H}om}_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}}\left(\nu^{-1}\mathcal{T}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)\\ &\cong \underline{\mathcal{H}om}_{\nu^{-1}\mathcal{O}}\left(\nu^{-1}\mathcal{T}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)\\ &\cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^1.\end{aligned}$$

This is $\nabla_{\mathrm{dR}}^{\dagger,+}$.

Remark 4.3.4. Definition 4.3.3 is inspired by the procedure that relates $\mathcal{D}$-module structures to connections in the classical theory.

Lemma 4.3.5. $\nabla_{\mathrm{dR}}^{\dagger,+}$ *fits into the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} & \xrightarrow{\nabla_{\mathrm{dR}}^{\dagger,+}} & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega \\ \uparrow & & \uparrow \\ \nu^{-1}\mathcal{O} & \xrightarrow{\nu^{-1}\nabla} & \nu^{-1}\mathcal{O}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega \end{array}$$

of sheaves of k-ind-Banach spaces.

Proof. $\nu^{-1}\mathcal{O} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ is $\nu^{-1}\widehat{\mathcal{D}}$-linear by Theorem 4.2.1. □

Lemma 4.3.6. *Assume that X is affinoid and equipped with an étale morphism $X \to \mathbb{T}^d$. The étale map $X \to \mathbb{T}^d$ furnishes an isomorphism $\Omega^1 \cong \bigoplus_{i=1}^d \mathcal{O}dT_i$. Together with Corollary 3.5.7, this gives the vertical morphisms in the diagrams*

$$\begin{array}{ccc} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U) & \xrightarrow{\nabla_{\mathrm{dR}}^{\dagger,+}(U)} & \bigoplus_{i=1}^d \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)dT \\ \cong\uparrow & & \cong\uparrow \\ \mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\widehat{\otimes}_{k_0}k_0\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle & \xrightarrow{\mathrm{id}\widehat{\otimes}d^\infty} & \mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\widehat{\otimes}_{k_0}\Omega_{k_0\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle/k_0} \end{array} \tag{4.3.1}$$

for every affinoid perfectoid $U \in X_{pro\acute{e}t}/\widetilde{X}$; d^∞ denotes the colimit of the differentials

$$d^q\colon k_0\left\langle\frac{Z_1,\dots,Z_d}{p^q}\right\rangle \to \Omega_{k_0\left\langle\frac{Z_1,\dots,Z_d}{p^q}\right\rangle/k_0}$$

along $q \to \infty$. These diagrams 4.3.1 commute.

Proof. Firstly, we clarify the definition of the diagram (4.3.1):

- The codomain of $\nabla_{\mathrm{dR}}^{\dagger,+}(U)$ is indeed

$$\bigoplus_{i=1}^d \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)dT \cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\widehat{\otimes}_{\nu^{-1}\mathcal{O}(U)}\nu^{-1}\Omega^1(U) \cong \left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^1\right)(U).$$

- The vertical morphism at the left of (4.3.1) comes directly from Corollary 3.5.7 and Lemma 2.4.7.

- The vertical morphism at the right-hand side is the composition

$$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\widehat{\otimes}_{k_0}\Omega_{k_0\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle/k_0} \cong \bigoplus_{i=1}^d\left(\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U)\widehat{\otimes}_{k_0}k_0\left\langle\frac{Z_1,\dots,Z_d}{p^\infty}\right\rangle\right) \cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}(U),$$

  where the morphism in the right comes again from Corollary 3.5.7 and Lemma 2.4.7.

$\nabla^{\dagger,+}_{\mathrm{dR}}(U)$ is, by the definition of ind-completions, the colimit of bounded maps

$$\nabla^{q,q',+}_{\mathrm{dR}}(U)\colon\ \mathbb{B}^{q,+}_{\mathrm{dR}}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^q}\right\rangle\to\bigoplus_{i=1}^{d}\mathbb{B}^{q',+}_{\mathrm{dR}}(U)\left\langle\frac{Z_1,\ldots,Z_d}{p^{q'}}\right\rangle dT_i$$

along pairs of natural numbers $q\leq q'$. These $\nabla^{q,q',+}_{\mathrm{dR}}(U)$ are connections in the following sense: Firstly, they are, by definition, $\mathbb{B}^{q,+}_{\mathrm{dR}}(U)$-linear. Secondly, they satisfy the Leibniz rule because the sections of $\mathcal{T}$ satisfy the Leibniz rule. Lastly, they vanish on $\mathbb{B}^{q,+}_{\mathrm{dR}}(U)$: By definition, $\nabla^{q,q',+}_{\mathrm{dR}}$ sends a section $b\in\mathbb{B}^{q,+}_{\mathrm{dR}}(U)$ to the morphism $\nu^{-1}\mathcal{T}\to\mathbb{B}^{q',+}_{\mathrm{dR}}\left\langle\frac{Z_1,\ldots,Z_d}{p^{q'}}\right\rangle$, $P\mapsto bP(1)$ of sheaves of $\nu^{-1}\mathcal{O}$-ind-Banach modules on a localisation of $X_{\text{proét}}$. But this morphism is zero, again because the sections of $\mathcal{T}$ are derivations, that is $P(1)=0$.

Thus, to show the commutativity of the diagram (4.3.1), it remains to compute $\nabla^{q,q',+}_{\mathrm{dR}}(U)(Z_i)$ for all $i=1,\ldots,d$. This follows from Definition 4.3.3, using that $\{dT_i\}_{i=1,\ldots,d}$ is the $\mathcal{O}(U)$-basis of $\Omega^1(U)$ dual to $\{\partial_i\}_{i=1,\ldots,d}$. □

Lemma 4.3.7. *For every $i\geq 0$, there is a unique morphism*

$$\nabla^{+,i}_{\mathrm{dR}}\colon\ \mathcal{O}\mathbb{B}^{+}_{\mathrm{dR}}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^{i}\to\mathcal{O}\mathbb{B}^{+}_{\mathrm{dR}}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^{i+1}$$

of sheaves of $\mathbb{B}^{+}_{\mathrm{dR}}$-ind-Banach modules satisfying

$$\nabla^{+,i}_{\mathrm{dR}}(f\,d\omega_1\wedge\cdots\wedge d\omega_i)=\nabla^{+}_{\mathrm{dR}}(f)\wedge d\omega_1\wedge\cdots\wedge d\omega_i$$

for local sections $f\in\mathcal{O}\mathbb{B}^{+}_{\mathrm{dR}}$ and $\omega\in\nu^{-1}\Omega^{i}$. In particular, $\nabla^{+,0}_{\mathrm{dR}}=\nabla^{+}_{\mathrm{dR}}$ and $\nabla^{+,i+1}_{\mathrm{dR}}\circ\nabla^{+,i}_{\mathrm{dR}}=0$ for all $i\geq 0$.

Proof. Thanks to Lemma 4.3.6, we can proceed as in the classical situation, cf. for example [59, Tag 0FKF]. □

Theorem 4.3.8 (Poincaré Lemma)**.** *Lemma 4.3.7 gives rise to the de Rham complex*

$$0\longrightarrow\mathcal{O}\mathbb{B}^{+}_{\mathrm{dR}}\xrightarrow{\nabla^{+}_{\mathrm{dR}}}\mathcal{O}\mathbb{B}^{+}_{\mathrm{dR}}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^{1}\xrightarrow{\nabla^{+,1}_{\mathrm{dR}}}\cdots\xrightarrow{\nabla^{+,d-1}_{\mathrm{dR}}}\mathcal{O}\mathbb{B}^{+}_{\mathrm{dR}}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^{d}\to 0.$$

where $d:=\dim X$. It is strictly exact everywhere, except in degree zero: Here

$$\mathbb{B}^{+}_{\mathrm{dR}}\xrightarrow{\cong}\ker\nabla^{+}_{\mathrm{dR}}$$

is an isomorphism of sheaves of $\mathbb{B}^{+}_{\mathrm{dR}}$-ind-Banach algebras.

Proof. We may assume that X is affinoid and equipped with an étale morphism $X \to \mathbb{T}^d$, giving rise to the pro-étale covering $\widetilde{X} \to X$. We have to check that

$$\begin{aligned} 0 \longrightarrow \mathbb{B}_{\mathrm{dR}}^{+}(U) \longrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}(U) \longrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}(U) \widehat{\otimes}_{\nu^{-1}\mathcal{O}(U)} \nu^{-1}\Omega^{1}(U) \\ \longrightarrow \ldots \longrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}(U) \widehat{\otimes}_{\nu^{-1}\mathcal{O}(U)} \nu^{-1}\Omega^{d}(U) \longrightarrow 0 \end{aligned} \tag{4.3.2}$$

is strictly exact for every affinoid perfectoid $U \in X_{\text{proét}}/\widetilde{X}$, cf. Lemma B.1.3(iii). Corollary 3.5.7 and Lemma 4.3.6 imply that the complex (4.3.2) is isomorphic to

$$\varinjlim_{q \in \mathbb{N}} \mathbb{B}_{\mathrm{dR}}^{q,+}(U) \widehat{\otimes}_{k_0} \left(0 \to k_0 \to A^q \to \Omega^1_{A^q/k_0} \to \cdots \to \Omega^d_{A^q/k_0} \to 0\right). \tag{4.3.3}$$

where $A^q := k_0 \left\langle \frac{Z_1,\ldots,Z_d}{p^q} \right\rangle$. The complexes at the right-hand side, that is

$$0 \to k_0 \to A^q \to \Omega^1_{A^q/k_0} \to \cdots \to \Omega^d_{A^q/k_0} \to 0, \tag{4.3.4}$$

are by Lemma 4.3.6 the concatenations of the maps $k_0 \to A^q$ with the de Rham complexes of the affinoid algebras A_q. (4.3.4) is not exact, but the colimit

$$\varinjlim_{q \in \mathbb{N}} \left(0 \to k_0 \to A^q \to \Omega^1_{A^q/k_0} \to \cdots \to \Omega^d_{A^q/k_0} \to 0\right)$$

is strictly exact: this follows because the underlying complex of abstract k_0-vector spaces is exact by [15, remark following the proof of Corollary 1.3.3], thus the complex of complete bornological spaces is strictly exact by a version of the open mapping theorem, see [8, Theorem 4.9]. Apply Lemma 2.2.12 to see that the complex is strictly exact as a complex of k-ind-Banach spaces. Since $\mathbb{B}_{\mathrm{dR}}^{q,+}(U)\widehat{\otimes}_{k_0}-$ is exact, cf. Corollary 2.2.6. This implies that (4.3.3) is strictly exact as well. □

Chapter 5

Solution and de Rham functors

We use Schneiders' framework for homological algebra within quasi-abelian categories, cf. [52, section 1]. Fix the notation introduced in subsection 3.1. X is a smooth rigid-analytic k-variety of dimension $\dim X$.

5.1 Solution and de Rham functors for $\widehat{\mathcal{D}}$-modules

Equip $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ with the $\nu^{-1}\widehat{\mathcal{D}}$-$\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$ bimodule structure from Theorem 4.2.1.

$$\begin{aligned}\mathrm{Sol}^{+}\colon\ \mathbf{D}\left(\widehat{\mathcal{D}}\right)^{\mathrm{op}} &\to \mathbf{D}\left(\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right),\\ \mathcal{M}^{\bullet} &\mapsto \mathrm{R}\,\underline{\mathcal{H}om}_{\nu^{-1}\widehat{\mathcal{D}}}\left(\nu^{-1}\mathcal{M}^{\bullet}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)\end{aligned}$$

is the *positive solution functor*.

Remark 5.1.1. ν^{-1} is strongly exact by [18, the discussion following Lemma 2.26].

We refer to [18, subsection 5.2] for the definition of the duality functor

$$\mathbb{D}\colon\ \mathbf{D}\left(\widehat{\mathcal{D}}\right) \to \mathbf{D}\left(\widehat{\mathcal{D}}\right)^{\mathrm{op}}.$$

Following the classical [35, Proposition 4.2.1], the *positive de Rham functor* is

$$\mathrm{dR}^{+}\colon\ \mathbf{D}\left(\widehat{\mathcal{D}}\right) \to \mathbf{D}\left(\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right), \mathcal{M}^{\bullet} \mapsto \mathrm{Sol}^{+}\left(\mathbb{D}\left(\mathcal{M}^{\bullet}\right)\right)[\dim X]. \tag{5.1.1}$$

We compute some values of the solution and de Rham functors.

Definition 5.1.2. A sheaf of $\mathcal{O}$-ind-Banach modules is *locally finite free* if it is, locally, isomorphic to a finite direct sum of copies of $\mathcal{O}$ as a sheaf of $\mathcal{O}$-ind-Banach modules.

Definition 5.1.3. Let $\mathcal{N}$ denote an $\mathcal{O}$-module with integrable connection $\nabla_{\mathcal{N}}$. This induces a connection

$$\nabla\colon \nu^{-1}\mathcal{N}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+} \to \nu^{-1}\mathcal{N}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{1}\Omega^{1}$$

defined by the formula $\nabla := (\nu^{-1}\nabla_{\mathcal{N}})\,\widehat{\otimes}\,\mathrm{id} + \mathrm{id}\,\widehat{\otimes}\nabla_{\mathrm{dR}}^{\dagger,+}$. We write

$$\left(\nu^{-1}\mathcal{N}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)^{\nabla=0} := \ker\nabla.$$

Proposition 5.1.4. *Let $\mathcal{M}$ denote a $\widehat{\mathcal{D}}$-ind-Banach module which is locally finite free as an $\mathcal{O}$-module.* $\mathrm{Sol}^{+}(\mathcal{M})$ *is concentrated in degree* 0 *and*

$$\mathrm{H}^{0}\left(\mathrm{Sol}^{+}(\mathcal{M})\right) \cong \left(\nu^{-1}\,\mathbb{D}\,(\mathcal{M})\,\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)^{\nabla=0}.$$

We need the following result in order to prove Proposition 5.1.4.

Lemma 5.1.5. *The functor $-\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\widehat{\mathcal{D}}$ is strongly exact.*

Proof. We may work locally to assume that X is a smooth rigid analytic variety equipped with an étale morphism $X \to \mathbb{T}^{d}$. Fix the notations $\varprojlim^{\mathrm{b}}$ and $\widehat{\otimes}^{\mathrm{b}}$, cf. the end of section 2.2. In the following, the first isomorphism comes from [4, Lemma 3.4] and the fourth isomorphism comes from [18, Corollary 3.40]; see also *loc. cit.* the proof of Lemma 4.5.

$$\begin{aligned}
\widehat{\mathcal{D}}(X) &\cong \varprojlim_{r} \mathcal{O}\widehat{\otimes}_{k}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle \\
&\overset{2.2.11}{\cong} \varprojlim_{r}{}^{\mathrm{b}}\mathcal{O}\widehat{\otimes}_{k}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle \\
&\overset{2.2.13}{\cong} \varprojlim_{r}{}^{\mathrm{b}}\mathcal{O}\widehat{\otimes}_{k}^{\mathrm{b}}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle \\
&\cong \mathcal{O}\widehat{\otimes}_{k}^{\mathrm{b}}\varprojlim_{r}{}^{\mathrm{b}}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle \\
&\overset{2.2.13}{\cong} \mathcal{O}\widehat{\otimes}_{k}\varprojlim_{r}{}^{\mathrm{b}}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle \\
&\overset{2.2.11}{\cong} \mathcal{O}\widehat{\otimes}_{k}\varprojlim_{r}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle.
\end{aligned}$$

To show flatness, we work on the site $X_{\text{proét,affperfd}}^{\text{fin}}$, cf. Lemma 3.2.6. Lemma B.1.2 then computes the sections of the sheaf

$$\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\widehat{\mathcal{D}} \cong \mathcal{M}\widehat{\otimes}_{k}\varprojlim_{r}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle$$

as follows: they are $\mathcal{M}(U)\widehat{\otimes}_{k}\varprojlim_{r}k\langle p^{r}\partial_{1},\dots,p^{r}\partial_{d}\rangle$ over any affinoid perfectoid U. The result thus follows from Lemma 2.2.14. □

Proof of Proposition 5.1.4. First, we recall [18, Proposition 5.2], which gives a functorial isomorphism

$$\mathbb{D}(\mathcal{M}) \cong \underline{\mathcal{H}om}_{\mathcal{O}}(\mathcal{M}, \mathcal{O}). \tag{5.1.2}$$

Next, $S^\bullet \to \mathcal{O}$ denotes the Spencer resolution, cf. [18, Theorem 4.12]. *Loc. cit.* introduces $S^\bullet$ as a complex of sheaves of complete bornological k-vector spaces, and it establishes the following stronger fact: the complexes

$$\cdots \to S^{-2}(U) \to S^{-1}(U) \to S^0(U) \to \mathcal{O}(U) \to 0$$

are strictly exact for every affinoid subdomain $U \subseteq X$. Thus Lemma 2.2.12 applies, showing that $S^\bullet \to \mathcal{O}$ is a strictly exact complex of sheaves of k-ind-Banach spaces.

We recall the definition: We have $S^{-i} = \widehat{\mathcal{D}} \widehat{\otimes}_{\mathcal{O}} \wedge^i \mathcal{T}$ for all $i \in \mathbb{N}$, and $S^{-i} = 0$ for all $i < 0$. Here, $\mathcal{T}$ denotes the tangent sheaf, cf. Definition 4.3.2. The Spencer resolution is a locally free resolution of $\widehat{\mathcal{D}}$-ind-Banach modules. It is thus a resolution by strongly flat $\mathcal{O}$-ind-Banach modules, Since ν^{-1} is strongly exact, cf. [18, the discussion following Lemma 2.26], $\nu^{-1}S^\bullet \to \nu^{-1}\mathcal{O}$ is a locally free solution of $\nu^{-1}\widehat{\mathcal{D}}$-ind-Banach modules. Lemma 5.1.5 implies that it is a resolution by strongly flat $\nu^{-1}\mathcal{O}$-modules.

In the following, we freely use that $\mathcal{M}$ is finite locally free as an $\mathcal{O}$-module.

$$\begin{aligned}
&\mathrm{Sol}^+(\mathcal{M}) \\
&= \mathrm{R}\,\underline{\mathcal{H}om}_{\nu^{-1}\widehat{\mathcal{D}}}\left(\nu^{-1}\mathcal{M}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right) \\
&= \mathrm{R}\,\underline{\mathcal{H}om}_{\nu^{-1}\widehat{\mathcal{D}}}\left(\nu^{-1}\mathcal{M}\widehat{\otimes}^{\mathrm{L}}_{\nu^{-1}\mathcal{O}}\nu^{-1}\mathcal{O}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right) \\
&\cong \mathrm{R}\,\underline{\mathcal{H}om}_{\nu^{-1}\widehat{\mathcal{D}}}\left(\nu^{-1}\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}S^\bullet, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right) \\
&\overset{\text{B.1.6}}{\cong} \mathrm{R}\,\underline{\mathcal{H}om}_{\nu^{-1}\widehat{\mathcal{D}}}\left(\nu^{-1}\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\left(\nu^{-1}\widehat{\mathcal{D}}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\wedge^\bullet \mathcal{T}\right), \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right) \\
&\cong \mathrm{R}\,\underline{\mathcal{H}om}_{\nu^{-1}\mathcal{O}}\left(\nu^{-1}\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\wedge^\bullet \mathcal{T}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right).
\end{aligned} \tag{5.1.3}$$

$\nu^{-1}\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\wedge^\bullet \mathcal{T}$ is a complex of locally finite free $\nu^{-1}\mathcal{O}$-modules. Thus it computes the $\mathrm{R}\,\underline{\mathcal{H}om}$. Now continue with the computation:

$$\begin{aligned}
&\mathrm{Sol}^+(\mathcal{M}) \\
&\overset{(5.1.3)}{\cong} \underline{\mathcal{H}om}_{\nu^{-1}\mathcal{O}}\left(\nu^{-1}\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\wedge^\bullet \mathcal{T}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right) \\
&\overset{(5.1.2)}{\cong} \nu^{-1}\mathbb{D}(\mathcal{M})\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^\bullet\right).
\end{aligned} \tag{5.1.4}$$

$\mathbb{D}(\mathcal{M})$ is locally finite free as an $\mathcal{O}$-module by (5.1.2). Therefore, $\nu^{-1}\mathbb{D}(\mathcal{M})$ is strongly flat as a $\nu^{-1}\mathcal{O}$-ind-Banach module. It follows from (5.1.4) and Theorem 4.3.8

that $\mathrm{Sol}^+(\mathcal{M})$ is concentrated in degree 0. More precisely,

$$\begin{aligned}
&\mathrm{H}^0\left(\mathrm{Sol}^+(\mathcal{M})\right)\\
&\overset{(5.1.4)}{\cong} \mathrm{H}^0\left(\nu^{-1}\mathbb{D}(\mathcal{M})\,\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\left(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^\bullet\right)\right)\\
&= \left(\nu^{-1}\mathbb{D}(\mathcal{M})\,\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)^{\nabla=0}.
\end{aligned}$$

□

Corollary 5.1.6. *Let $\mathcal{M}$ denote a $\widehat{\mathcal{D}}$-ind-Banach module which is locally finite free as an $\mathcal{O}$-module. Then $\mathrm{dR}^+(\mathcal{M})$ is concentrated in degree $-\dim X$ and*

$$\mathrm{H}^{-\dim X}\left(\mathrm{dR}^+(\mathcal{M})\right) \cong \left(\nu^{-1}\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)^{\nabla=0}.$$

Proof. $\mathbb{D}(\mathcal{M})$ is again locally finite free as an $\mathcal{O}$-module, cf. [18, Proposition 5.2]. Thus Proposition 5.1.4 applies, showing that $\mathrm{dR}^+(\mathcal{M})$ is concentrated in degree $-\dim X$. Furthermore, we see with $\mathbb{D}^2(\mathcal{M}) \cong \mathcal{M}$, by [18, Theorem 7.17],

$$\begin{aligned}
\mathrm{H}^{-\dim X}\left(\mathrm{dR}^+(\mathcal{M})\right) &= \mathrm{H}^0\left(\mathrm{Sol}^+(\mathbb{D}(\mathcal{M}))\right)\\
&\cong \left(\nu^{-1}\mathbb{D}^2(\mathcal{M})\,\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)^{\nabla=0}\\
&\cong \left(\nu^{-1}\mathcal{M}\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)^{\nabla=0}.
\end{aligned}$$

□

Example 5.1.7. $\mathrm{dR}^+(\mathcal{O})$ is concentrated in degree $-\dim X$, where its cohomology is isomorphic to $\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$. This follows from Corollary 5.1.6 and a version of Theorem 4.3.8 for $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}$. See also Remark 4.3.1.

5.2 Compatibility with Scholze's functor I: period sheaves

We would like to compare our constructions to [54, section 7]. *Loc. cit.* works with the de Rham sheaves $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{dR}}^+$, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$, and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$; we explain that they carry canonical algebra structures from their overconvergent counterparts.

5.2.1 Relative period rings

We recall constructions from [54, section 6]. Consider the completion K of an algebraic extension of k which is perfectoid. Pick a ring of integral elements $K^+ \subseteq K$ containing k° and fix an affinoid perfectoid $(K, K+)$-algebra (R, R^+). Consider

Fontaine's map $\theta_{\inf}\colon \mathbb{A}_{\inf}(R, R^+) \to R^+$ and invert p to get $\vartheta_{\inf}\colon \mathbb{B}_{\inf}(R, R^+) \to R$. The *relative positive de Rham period ring* is

$$\mathbb{B}_{\mathrm{dR}}^{+}\left(R, R^+\right) := \varprojlim_{j \in \mathbb{N}} \mathbb{B}_{\inf}\left(R, R^+\right) / \left(\ker \vartheta_{\inf}\right)^j . \tag{5.2.1}$$

If K admits a compatible sequence of primitive pth roots of unity $\epsilon \in K^\flat$,

$$\mathbb{B}_{\mathrm{dR}}\left(R, R^+\right) := \mathbb{B}_{\mathrm{dR}}^{+}\left(R, R^+\right)[1/t] \tag{5.2.2}$$

is the *relative de Rham period ring.* Here, t is as in Definition 3.3.26. We aim to relate the period rings above to the overconvergent ones.

Notation 5.2.1. $\mathbb{A}_{\inf}^{(p)}(R, R^+)$ is the seminormed $W(\kappa)$-algebra $\mathbb{A}_{\inf}(R, R^+)$, equipped with the p-adic seminorm. It is a Banach algebra by Lemma 3.3.1.

Lemma 5.2.2. $\mathbb{A}_{\inf}^{(p)}(R, R^+) / (\ker \theta_{\inf})^j$ *is complete for every* $j \in \mathbb{N}$*.*

Proof. By Lemma 3.3.2, $\ker \theta_{\inf} = (\xi)$. Lemma 3.3.5(ii) implies

$$\|a\xi^j\| = \|a\| \tag{5.2.3}$$

for all $a \in \mathbb{A}_{\inf}^{(p)}(R, R^+)$. We show that the ideal (ξ^j) is closed. Consider a sequence $(f_i\xi^j)_{i\in\mathbb{N}} \subseteq (\xi^j)$ which converges to an element h. But then (f_i) is Cauchy:

$$\|f_i - f_{i+1}\| \overset{(5.2.3)}{=} \|f_i\xi^j - f_{i+1}\xi^j\| \to 0 \text{ for } i \to \infty,$$

and $h = \lim_{i\to\infty}(f_i\xi^j) = (\lim_{i\to\infty} f_i)\,\xi^j \in (\xi^j)$ follows. □

Notation 5.2.3. $\mathbb{B}_{\inf}^{(p)}(R, R^+)$ is the seminormed k_0-algebra $\mathbb{B}_{\inf}(R, R^+)$ with unit ball $\mathbb{A}_{\inf}(R, R^+)$. It is a Banach algebra by Lemma 3.3.1.

Lemma 5.2.4. $\mathbb{B}_{\inf}^{(p)}(R, R^+) / (\ker \vartheta_{\inf})^j$ *is complete for every* $j \in \mathbb{N}$*.*

Proof. By Lemma 2.3.6,

$$\mathbb{A}_{\inf}^{(p)}\left(R, R^+\right) \widehat{\otimes}_{W(\kappa)} k_0 \xrightarrow{\cong} \mathbb{B}_{\inf}^{(p)}\left(R, R^+\right).$$

By Lemma 2.3.7, which applies because $\mathbb{A}_{\inf}^{(p)}(R, R^+)$ does not have p-torsion by Lemma 3.3.1, the result would follow once

$$\mathbb{A}_{\inf}^{(p)}\left(R, R^+\right) \xrightarrow{\xi^j} \mathbb{A}_{\inf}^{(p)}\left(R, R^+\right) \tag{5.2.4}$$

is a strict monomorphism. It is injective by Lemma 3.3.2, has closed image by Lemma 5.2.2 and is open onto its image because $\|a\xi\| = \|a\|$ for every $a \in \mathbb{A}_{\inf}^{(p)}(R, R^+)$, cf. Lemma 3.3.5(ii). (5.2.4) is thus a strict monomorphism by Lemma 2.1.7(iii). □

$\mathbb{A}_{\inf}(R, R^+)$ still carries the (p, ξ)-adic topology. Consider

$$\mathbb{A}_{\inf}\left(R, R^+\right) \to \mathbb{A}_{\inf}^{(p)}\left(R, R^+\right) / (\ker \theta_{\inf})^j .$$

It map is a morphism of $W(\kappa)$-Banach algebras by Lemma 2.1.3. It lifts to a morphism

$$\mathbb{B}_{\mathrm{dR}}^{q,+}\left(R, R^+\right) \to \mathbb{B}_{\inf}^{(p)}\left(R, R^+\right) / (\ker \vartheta_{\inf})^j$$

of k_0-Banach algebras for every $q \in \mathbb{N}$. Now compute the inverse limit along j to get

$$|\, \mathbb{B}_{\mathrm{dR}}^{q,+}\left(R, R^+\right) | \to \mathbb{B}_{\mathrm{dR}}\left(R, R^+\right),$$

where $|-|$ refers to the underlying abstract ring, cf. (2.2.1). Finally, we get the following morphism of k_0-algebras:

$$|\, \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(R, R^+\right) | = \varinjlim_{q \in \mathbb{N}} |\, \mathbb{B}_{\mathrm{dR}}^{q,+}\left(R, R^+\right) | \to \mathbb{B}_{\mathrm{dR}}^{+}\left(R, R^+\right).$$

Similarly, $\mathbb{B}_{\mathrm{dR}}(R, R^+)$ is canonically a $|\, \mathbb{B}_{\mathrm{dR}}^{\dagger}(R, R^+)\,|$-algebra via

$$|\, \mathbb{B}_{\mathrm{dR}}^{\dagger}\left(R, R^+\right) | = \varinjlim_{t\times} \varinjlim_{q \in \mathbb{N}} |\, \mathbb{B}_{\mathrm{dR}}^{q,+}\left(R, R^+\right) | \to \varinjlim_{t\times} \mathbb{B}_{\mathrm{dR}}^{+}\left(R, R^+\right) \cong \mathbb{B}_{\mathrm{dR}}\left(R, R^+\right).$$

5.2.2 Period sheaves

We refer the reader to [54, Definition 6.1] for the definition of the sheaf $\mathbb{B}_{\mathrm{dR}}^{+}$ on $X_{\text{proét}}$. By *loc. cit.* Theorem 6.5(ii), its sections over an affinoid perfectoid affinoid perfectoid $U \in X_{\text{proét}}$ with $\widehat{U} = \operatorname{Spa}(R, R^+)$ are

$$\mathbb{B}_{\mathrm{dR}}^{+}(U) = \mathbb{B}_{\mathrm{dR}}^{+}\left(R, R^+\right).$$

Recall Definition 2.6.3. Lemma 2.6.4 and Theorem 3.4.2 give

$$|\, \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,|(U) = |\, \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\left(R, R^+\right) |.$$

From the discussion in subsection 5.2.1, we find a morphism of sheaves

$$|\, \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,||_{X^{\mathrm{fin}}_{\text{proét,affperfd}}} \to \mathbb{B}_{\mathrm{dR}}^{+}|_{X^{\mathrm{fin}}_{\text{proét,affperfd}}}$$

of k_0-algebras. Apply Lemma 3.2.6 to extend it to

$$|\, \mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,| \to \mathbb{B}_{\mathrm{dR}}^{+},$$

which is a morphism of sheaves of k_0-algebras on $X_{\text{proét}}$. Similarly, one finds

$$|\, \mathbb{B}_{\mathrm{dR}}^{\dagger}\,| \to \mathbb{B}_{\mathrm{dR}},$$

a canonical morphism of sheaves of k_0-algebras on $X_{\text{proét}}$.

5.2.3 Period structure sheaves

Notation 5.2.5. For any two $W(\kappa)$-algebras R and S, $R\widehat{\otimes}^{(p)}_{W(\kappa)}S$ is the p-adic completion of $R \otimes_{W(\kappa)} S$. We equip it with the p-adic seminorm.

Recall the following definition from [55].

Definition 5.2.6. $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ is the sheafification of the presheaf

$$\mathcal{O}\mathbb{B}^{+,\mathrm{psh}}_{\mathrm{dR}} \colon U = \text{“}\varinjlim_{i\in I}\text{”} U_i \mapsto \varinjlim_{i\geq i_0} \varprojlim_{j\in\mathbb{N}} \left(\mathcal{O}^+(U_i)\widehat{\otimes}^{(p)}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U)\right)[1/p] \Big/ (\ker \mathcal{O}\vartheta_{\mathrm{inf}})^j$$

of abstract k-algebras on $X_{\mathrm{proét}}$. Here, $\mathcal{O}\vartheta_{\mathrm{inf}}$ denotes the map

$$\mathcal{O}\theta_{\mathrm{inf}}[1/p] \colon \left(\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U)\right)[1/p] \to \widehat{\mathcal{O}}^+(U)[1/p] = \widehat{\mathcal{O}}(U).$$

$\mathcal{O}\theta_{\mathrm{inf}}$ has been defined at the beginning of section 3.5.

In this subsection, we construct of a canonical morphism $|\,\mathcal{O}\mathbb{B}^{\dagger,+}_{\mathrm{dR}}\,| \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$. Note that $\theta_{\mathrm{inf}} \colon \mathbb{B}_{\mathrm{inf}} \to \widehat{\mathcal{O}}$ extends canonically to a morphism $\theta_{\mathrm{dR}} \colon \mathbb{B}^+_{\mathrm{dR}} \to \widehat{\mathcal{O}}$.

Notation 5.2.7. Let U be an affinoid perfectoid with $\widehat{U} = \mathrm{Spa}\,(R, R^+)$. Then

$$\begin{aligned}&\left(\mathbb{B}^+_{\mathrm{dR}}(U)/\left((\ker\vartheta_{\mathrm{dR}})^j\right)\right)[\![Z_1,\ldots,Z_d]\!]/\left(Z_1,\ldots,Z_d\right)^j\\ &\cong \bigoplus_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|<j}} \left(\mathbb{B}^{(p)}_{\mathrm{inf}}\left(R,R^+\right)/\left((\ker\vartheta_{\mathrm{inf}})^j\right)\right) Z^\alpha\end{aligned}$$

is viewed as a Banach space with the coproduct norm, cf. Lemma 5.2.4.

Lemma 5.2.8. *Assume that X is affinoid and equipped with an étale morphism $X \to \mathbb{T}^d$. Let $U \in X_{\text{proét}}$ be affinoid perfectoid. Fix a pro-étale presentation $U = \text{”}\varinjlim_{i\in I}\text{”}U_i$ and fix $i_0 \in I$ as in Notation 3.5.4. Let $i \geq i_0$ be arbitrary. Then*

$$\begin{aligned}&\left(\mathbb{B}^+_{\mathrm{dR}}(U)/\left((\ker\vartheta_{\mathrm{dR}})^j\right)\right)[\![Z_1,\ldots,Z_d]\!]/\left(Z_1,\ldots,Z_d\right)^j\\ &\xrightarrow{\cong} \left(\mathcal{O}^+(U_i)\widehat{\otimes}^{(p)}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U)\right)[1/p]\Big/(\ker\mathcal{O}\vartheta_{\mathrm{inf}})^j\end{aligned}$$

given by $Z_l \mapsto u_l$, cf. (3.5.1), are isomorphisms of seminormed k-vector spaces for every $j \in \mathbb{N}$. In particular, the right-hand sides are k-Banach spaces.

Proof. See the [54, proof of Proposition 6.10] for the construction of the inverse map. One checks that it is bounded. Thus [54] implies that it is not merely an isomorphism of abstract vector spaces, but an isomorphism of seminormed vector spaces. □

We continue to assume that X is affinoid and equipped with an étale morphism $X \to \mathbb{T}^d$. Let $U = \text{“}\varprojlim\text{”}_{i\in I} U_i \in X_{\text{proét}}$ be affinoid perfectoid. $\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\inf}(U)$ carries the (p,ξ)-adic topology. Consider

$$\mathcal{O}^+(U_i)\widehat{\otimes}_{W(\kappa)}\mathbb{A}_{\inf}(U) \to \left(\mathcal{O}^+(U_i)\widehat{\otimes}^{(p)}_{W(\kappa)}\mathbb{A}_{\inf}(U)\right)[1/p]\Big/(\ker \mathcal{O}\theta_{\inf})^j\,, \tag{5.2.5}$$

where the codomain carries the quotient seminorm. We find with Lemma 5.2.8 that the morphism (5.2.8) lifts to a morphism

$$|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+,\dagger,\mathrm{psh}}(U)\,| \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}(U)\,,$$

after taking the colimit along $i \in I$. By the Remark 3.5.8, this gives

$$|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+,\dagger}(U)\,| \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}(U)\,.$$

Lemma 2.6.4 implies that we have constructed a morphism

$$|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,||_{X^{\mathrm{fin}}_{\text{proét,affperfd}}} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}\,|_{X^{\mathrm{fin}}_{\text{proét,affperfd}}}$$

of sheaves of abstract k-algebras. Apply Lemma 3.2.6 to get

$$|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,| \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+};$$

it is by construction a canonical morphism of sheaves of k-algebras. That is, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$ is canonically a sheaf of $|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,|$-algebras. We observe that this algebra structure is compatible with the $\nu^{-1}\mathcal{O}$-module structure. In particular, the vertical morphism at the right-hand side of the diagram (5.2.6) is well-defined.

Lemma 5.2.9. *Recall the Definition 4.3.3 of* $\nabla_{\mathrm{dR}}^{\dagger,+}$. *The* $\mathbb{B}_{\mathrm{dR}}^{+}$*-linear connection* ∇_{dR}^{+} *has been defined in [54, section 6]; it fits into the commutative diagram*

$$\begin{array}{ccc}
\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+} & \xrightarrow{\quad\nabla_{\mathrm{dR}}^{+}\quad} & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}\otimes_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^1 \\
\uparrow & & \uparrow \\
|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,| & \xrightarrow{|\nabla_{\mathrm{dR}}^{\dagger,+}|} & |\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,\widehat{\otimes}_{\nu^{-1}\mathcal{O}}\nu^{-1}\Omega^1| = |\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,|\otimes_{\nu^{-1}|\mathcal{O}|}\nu^{-1}|\Omega^1|.
\end{array} \tag{5.2.6}$$

Proof. This is clear from Lemma 4.3.6 and the definition of ∇_{dR}^{+}. □

Finally, we recall that the sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is obtained from $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$ by inverting t, locally on the pro-étale site, cf. [54, Definition 6.1(iv)]. The algebra structure $|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,| \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}$ thus induces a canonical $|\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger}\,|$-algebra structure on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$.

5.3 Compatibility with Scholze's functor II: conjecture

Definition 5.3.1. [54, Definition 7.4] A filtered $\mathcal{O}$-module with integrable connection is a locally free $\mathcal{O}$-module $\mathcal{E}$ on X, together with a separated and exhaustive decreasing filtration $\operatorname{Fil}^n \mathcal{E}$ by locally direct summands, and an integrable connection ∇ satisfying Griffiths transversality with respect to the filtration, that is $\mathcal{T} \cdot \operatorname{Fil}^n \mathcal{E} \subseteq \operatorname{Fil}^{n-1} \mathcal{E}$ for all $n \in \mathbb{Z}$. Here, $\mathcal{T}$ denotes the tangent sheaf, cf. Definition 4.3.2.

See [54, Definition 6.8] for the definition of the descending filtration on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$; its zeroth filtered piece is $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$. *Loc. cit.* also constructs a $\mathbb{B}_{\mathrm{dR}}$-linear connection $\nabla_{\mathrm{dR}} \colon \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\nu^{-1}\mathcal{O}} \nu^{-1}\Omega^1$. Now Scholze's functor [54, Theorem 7.6] is

$$\begin{aligned} \left\{ \begin{matrix} \text{filtered } \mathcal{O}\text{-modules} \\ \text{with integrable connection} \end{matrix} \right\} &\to \mathbf{Mod}\left(\mathbb{B}_{\mathrm{dR}}^+\right) \\ \mathcal{E} &\mapsto \operatorname{Fil}^0 \left(\nu^{-1}\mathcal{E} \otimes_{\nu^{-1}\mathcal{O}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}\right)^{\nabla=0}. \end{aligned} \tag{5.3.1}$$

We would like to compare Scholze's functor (5.3.1) to the positive de Rham functor 5.1.1 First, we compare their domains.

Lemma 5.3.2. *Consider an $\mathcal{O}$-module $\mathcal{E}$ with integrable connection.*

(i) Equipped with the trivial filtration, $\mathcal{E}$ is a filtered $\mathcal{O}$-module with integrable connection in the sense of Definition 5.3.1.

(ii) $\mathcal{E}$ is canonically a sheaf of $\widehat{\mathcal{D}}$-ind-Banach modules.

Proof. The trivial filtration is $\operatorname{Fil}^n \mathcal{E} := \mathcal{E}$ for $n \leq 0$ and $\operatorname{Fil}^n \mathcal{E} = 0$ otherwise. (i) is obvious. (ii) follows from [6, Theorem B]: *loc. cit.* allows to view $\mathcal{E}$ as a sheaf of abstract $\widehat{\mathcal{D}}$-modules, it is thus canonically a sheaf of complete bornological $\widehat{\mathcal{D}}$-modules by [18, Theorem 4.4]. One deduces from Lemma 2.2.12 that they are sheaves of k-ind-Banach spaces, thus they are sheaves of $\widehat{\mathcal{D}}$-ind-Banach modules by Lemma 2.2.13. □

Fix an $\mathcal{O}$-module $\mathcal{E}$ with integrable connection. View it as a filtered $\mathcal{O}$-module with integrable connection via Lemma 5.3.2(i). Scholze's functor (5.3.1) sends it to a sheaf $\mathcal{L}$ of $\mathbb{B}_{\mathrm{dR}}^+$-modules. Because $\mathcal{E}$ carries the trivial filtration,

$$\mathcal{L} = \operatorname{Fil}^0 \left(\nu^{-1}\mathcal{E} \otimes_{\nu^{-1}\mathcal{O}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}\right)^{\nabla=0} = \left(\nu^{-1}\mathcal{E} \otimes_{\nu^{-1}\mathcal{O}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+\right)^{\nabla=0}.$$

On the other hand, $\mathcal{E}$ is a complex of complete bornological $\widehat{\mathcal{D}}$-modules concentrated in degree zero, cf. Lemma 5.3.2(ii). Compute

$$|\operatorname{H}^{-\dim X}\left(\mathrm{dR}^+(\mathcal{E})\right)| = |\left(\nu^{-1}\mathcal{E}\widehat{\otimes}_{\nu^{-1}\mathcal{O}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\right)^{\nabla=0}| = \left(\nu^{-1}\mathcal{E} \otimes_{\nu^{-1}\mathcal{O}} |\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{\dagger,+}|\right)^{\nabla=0}$$

with Corollary 5.1.6. Note that the tensor product at the right-hand side does not have to be completed because $\nu^{-1}\mathcal{E}$ is a locally finite free sheaf of $\nu^{-1}\mathcal{O}$-modules. Now we find with Lemma 5.2.9 that there is a canonical morphism

$$|\,\mathrm{H}^{-\dim X}\left(\mathrm{dR}^{+}\left(\mathcal{E}\right)\right)| \to \mathcal{L}$$

of sheaves of $|\,\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,|$-modules. In particular, it factors through a morphism

$$|\,\mathrm{H}^{-\dim X}\left(\mathrm{dR}^{+}\left(\mathcal{E}\right)\right)| \otimes_{|\,\mathbb{B}_{\mathrm{dR}}^{\dagger,+}\,|} \mathbb{B}_{\mathrm{dR}}^{+} \xrightarrow{\cong} \mathcal{L} \tag{5.3.2}$$

of sheaves of $\mathbb{B}_{\mathrm{dR}}^{+}$-modules.

Conjecture 5.3.3. *This morphism (5.3.2) is an isomorphism for any $\mathcal{O}$-module with integrable connection $\mathcal{E}$.*

Appendix A

Closed symmetric monoidal categories

Consider a closed symmetric monoidal category $(\mathbf{C}, 1, \otimes)$.

Lemma A.0.1. *Let $R \in \mathbf{C}$ denote a monoid and M an R-module object. Fix an epimorphism $\phi\colon M \to N$. If there exists an R-module structure on N making ϕ R-linear, then this structure is unique.*

Proof. Consider two R-module structures on N with actions $\mathrm{a}_{N,1}\colon R \otimes N \to N$, respectively $\mathrm{a}_{N,2}$, and units $1_{N,1}\colon R \to N$, respectively $1_{N,2}$. We assume that ϕ is R-linear with respect to both R-module structures on N.

Denote the action of R on M by $\mathrm{a}_M\colon R \otimes M \to M$. The ϕ-linearities imply

$$\mathrm{a}_{N,1} \circ (\mathrm{id}_R \otimes \phi) = \phi \circ \mathrm{a}_M, \text{ and } \mathrm{a}_{N,2} \circ (\mathrm{id}_R \otimes \phi) = \phi \circ \mathrm{a}_M .$$

But $\mathrm{id}_R \otimes \phi$ is an epimorphism because $\mathbf{C}$ is closed; $\mathrm{a}_{N,1} = \mathrm{a}_{N,2}$ follows.

Furthermore, the ϕ-linearities imply $1_{N,1} = \phi \circ 1_M = 1_{N,2}$. □

The following lemma is well-known, thus we omit the proof.

Lemma A.0.2. *Let R, S denote two monoids in $\mathbf{C}$ Then $R \otimes S$ becomes a monoid as follows. The multiplication is the composition*

$$(R \otimes S) \otimes (R \otimes S) \cong (R \otimes R) \otimes (S \otimes S) \xrightarrow{\mu_R \otimes \mu_S} R \otimes S$$

and the unit is the composition

$$1 \cong 1 \otimes 1 \xrightarrow{1_R \widehat{\otimes} 1_S} R \otimes S.$$

Here μ_* is the multiplication and 1_* is the unit of $* \in \{R, S\}$.

Definition A.0.3. Let R and S denote two monoid objects in $\mathbf{C}$. An R-S-bimodule object M is an $R \otimes S^{\mathrm{op}}$-module object. Here, the underlying $\mathbf{C}$-objects of S and S^{op} coincide, but the multiplication is performed in the reverse order.

A morphism between two R-S-bimodule objects is R-S-linear if it is a morphism of $R \otimes S^{\mathrm{op}}$-module objects.

The following corollary is a special case of Lemma A.0.1.

Corollary A.0.4. *Let $R, S \in \mathbf{C}$ denote monoids. M is an R-S-bimodule object. Fix an epimorphism $\phi\colon M \to N$. If there exists an R-S-bimodule structure on N making ϕ R-S-linear, then this structure is unique.*

The following two lemma are easy to verify; we leave the details to the reader.

Lemma A.0.5. *Suppose $\mathbf{C}$ admits all limits. Consider the tower of monoids*

$$\cdots \to R_2 \to R_1 \to R_0;$$

here, the maps are multiplicative. Then $R := \varprojlim_{r\in\mathbb{N}} R$ becomes a monoid object:

$$R \otimes R \to \varprojlim_{r\in\mathbb{N}} (R_r \otimes R_r) \to \varprojlim_{r\in\mathbb{N}} R_r = R$$

is the multiplication and the unit is

$$1 = \varprojlim_{r\in\mathbb{N}} 1 \to \varprojlim_{r\in\mathbb{N}} R_r = R.$$

Lemma A.0.6. *Suppose $\mathbf{C}$ admits all colimits. Consider the tower of monoids*

$$S^0 \to S^1 \to S^2 \to \dots;$$

here, the maps are multiplicative. Then $:= \varinjlim_{q\in\mathbb{N}} S^q$ becomes a monoid object:

$$S \otimes S = \varinjlim_{q\in\mathbb{N}} (S^q \otimes S^q) \to \varinjlim_{q\in\mathbb{N}} S^q = S$$

is the multiplication and the unit is

$$1 = \varinjlim_{q\in\mathbb{N}} 1 \to \varinjlim_{q\in\mathbb{N}} S^q = S.$$

We utilise Lemma A.0.7 in the proofs of Theorem 4.2.1 and Corollary 4.2.10.

Lemma A.0.7. *Suppose* $\mathbf{C}$ *admits all limits and colimits and consider the towers*

$$\cdots \to R_2 \to R_1 \to R_0, \text{ and}$$
$$S^0 \to S^1 \to S^2 \to \dots$$

of monoids; here, the maps are multiplicative. Let

$$M^0 \to M^1 \to M^2 \to \dots$$

denote a tower of objects in $\mathbf{C}$. *We further have* R_r*-*S^q*-bimodule structures on* M^q *for all* $r \leq q$ *such that*

(i) the following diagrams commute for all $r' \geq r$*:*

$$\begin{array}{ccc} (R_r \otimes S^{q,\mathrm{op}}) \otimes M^q & \longrightarrow & M^q \\ \uparrow & & \| \\ (R_{r'} \otimes S^{q,\mathrm{op}}) \otimes M^q & \longrightarrow & M^q. \end{array}$$

Here, the horizontal maps denote the bimodule actions. Furthermore,

(ii) the following diagrams commute for all $q' \geq q$*:*

$$\begin{array}{ccc} 1 & \longrightarrow & M^{q'} \\ \| & & \uparrow \\ 1 & \longrightarrow & M^q. \end{array}$$

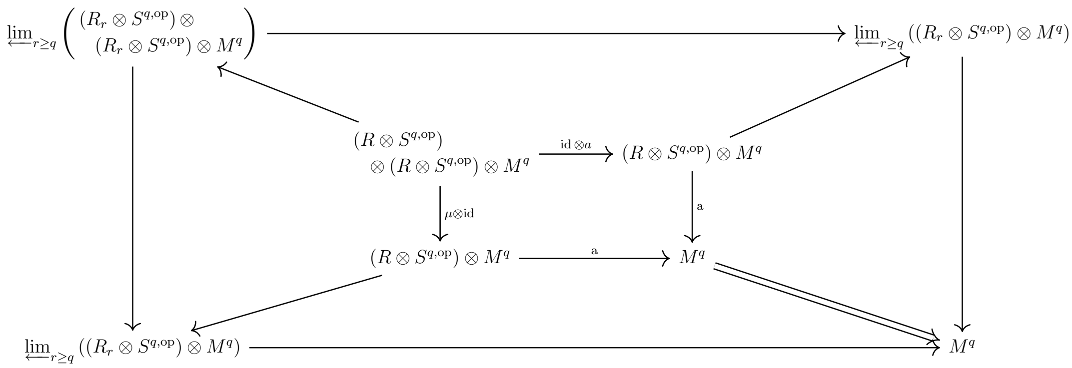
Here, the horizontal maps denote the unit maps.

Apply Lemma A.0.5 and A.0.6 to turn both $R := \varprojlim_{r\in\mathbb{N}} R_r$ *and* $S = \varinjlim_{q\in\mathbb{N}} S^q$ *into monoids. Then the composition*

$$\begin{aligned} (R \otimes S^{\mathrm{op}}) \otimes M &= \varinjlim_{q\in\mathbb{N}} (R \otimes S^{q,\mathrm{op}}) \otimes M^q \\ &\to \varinjlim_{q\in\mathbb{N}} \left(\left(\varprojlim_{r\geq q} R_r \right) \otimes S^{q,\mathrm{op}} \right) \otimes M^q \\ &\to \varinjlim_{q\in\mathbb{N}} \varprojlim_{r\geq q} (R_r \otimes S^{q,\mathrm{op}}) \otimes M^q \\ &\overset{(i)}{\to} \varinjlim_{q\in\mathbb{N}} M^q \\ &= M \end{aligned}$$

defines an R*-*S*-bimodule action on* $M := \varinjlim_{q\in\mathbb{N}} M^q$*. The unit* $1 \to M$ *is the colimit of the units* $1 \to M^q$*; this is well-defined by (ii).*

Proof. We denote multiplication maps by μ and module actions by a. Consider

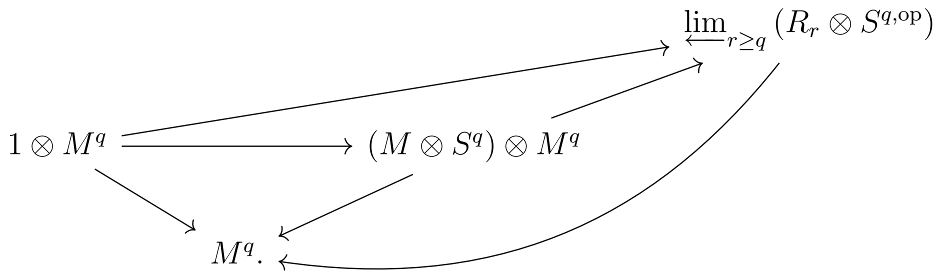


(A.0.1)

for every $q \in \mathbb{N}$. The diagram is commutative: this is obvious for all rectangles except the middle one. However, the commutativity of the middle rectangle follows from the commutativity of the others. Now consider again the square in the middle of (A.0.1) and pass to the colimit along $q \to \infty$ to get the commutative diagram

$$\begin{array}{ccc} (R \otimes S^{q,\mathrm{op}}) \otimes (R \otimes S^{q,\mathrm{op}}) \otimes M^q & \xrightarrow{\mathrm{id} \otimes \mathrm{a}} & (R \otimes S^{q,\mathrm{op}}) \otimes M^q \\ \downarrow{\scriptstyle \mu\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{a}} \\ (R \otimes S^{q,\mathrm{op}}) \otimes M^q & \xrightarrow{\quad\mathrm{a}\quad} & M^q \end{array}$$

It remains to check that the R-S-bimodule action preserves the unit map. For every $q \in \mathbb{N}$, consider the diagram



(A.0.2)

The diagram is commutative: this is obvious for all triangles except the one in the bottom left corner. However, the commutativity of the triangle follows from the commutativity of the others. Now consider again the triangle in the bottom left corner of (A.0.2) and pass to the colimit along $q \to \infty$ to get the diagram

$$\begin{array}{ccc} 1 \otimes M^q & \longrightarrow & (M \otimes S^q) \otimes M^q \\ & \searrow \quad \swarrow & \\ & M^q, & \end{array}$$

which is commutative. That is, the R-S-bimodule action preserves the unit map. $\square$

Appendix B

Sheaves valued in quasi-abelian categories

We assume the reader is familiar with quasi-abelian categories as developed by Schneiders [52]. We primarily cite [18], which works with sheaves on G-topological spaces, rather than topological spaces. This is still not the needed level of generality, but the discussions *loc. cit.* generalise to our setting. We prove a folklore result in subsection B.2 which we couldn't find in the literature.

B.1 Categories of sheaves

Fix a site X and a quasi-abelian category $\mathbf{E}$.

Definition B.1.1. An $\mathbf{E}$-*presheaf* or *presheaf with values in* $\mathbf{E}$ is a functor

$$\mathcal{F}\colon X^{\mathrm{op}} \to \mathbf{E}\,.$$

An $\mathbf{E}$-*sheaf* or *sheaf with values in* $\mathbf{E}$ is an $\mathbf{E}$-presheaf $\mathcal{F}$ such that for any open $U \in X$ and any covering $\mathfrak{U}$ of U,

- the products $\prod_{V\in\mathfrak{U}} \mathcal{F}(V)$ and $\prod_{W,W'\in\mathfrak{U}} \mathcal{F}(W \times_U W')$ exist, and
- $0 \to \mathcal{F}(U) \to \prod_{V\in\mathfrak{U}} \mathcal{F}(V) \to \prod_{W,W'\in\mathfrak{U}} \mathcal{F}(W \times_U W')$ is strictly exact.

Lemma B.1.2. *Suppose that X admits only finite coverings and consider a strongly left exact functor $\mathbf{F}\colon \mathbf{E}_1 \to \mathbf{E}_2$ between two quasi-abelian categories which admit all finite products. Then for any $\mathbf{E}_1$-sheaf $\mathcal{F}$, $\mathbf{F} \circ \mathcal{F}$ is a $\mathbf{E}_2$-sheaf.*

Proof. This is because $\mathbf{F}$ commutes with finite products, cf. [52, Remark 1.1.13]. □

From now on, $\mathbf{E}$ is elementary.

The morphisms in the category of presheaves $\mathrm{Psh}(X, \mathbf{E})$ are the natural transformations. The category of sheaves $\mathrm{Sh}(X, \mathbf{E})$ is the corresponding full subcategory. As in the classical theory, the inclusion $\mathrm{Psh}(X, \mathbf{E}) \hookrightarrow \mathrm{Sh}(X, \mathbf{E})$ has a left adjoint $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$. This is *sheafification*, cf. [18, the beginning of subsection 2.3].

Lemma B.1.3. *The categories of presheaves and sheaves are well-behaved:*

(i) $\mathrm{Psh}(X, \mathbf{E})$ *and* $\mathrm{Sh}(X, \mathbf{E})$ *are quasi-abelian categories.*

(ii) $\mathrm{Psh}(X, \mathbf{E})$ *and* $\mathrm{Sh}(X, \mathbf{E})$ *are complete and cocomplete. Limits in* $\mathrm{Sh}(X, \mathbf{E})$ *are the same as limits in* $\mathrm{Psh}(X, \mathbf{E})$ *and a colimit in* $\mathrm{Sh}(X, \mathbf{E})$ *is the sheafification of the colimit in* $\mathrm{Psh}(X, \mathbf{E})$*.*

(iii) Sheafification is strongly exact.

Now assume that $\mathbf{E}$ is closed symmetric monoidal with unit $1 \in \mathbf{E}$ and tensor product $\otimes$. This gives a symmetric monoidal structure on $\mathrm{Sh}(X, \mathbf{E})$ as follows. 1_X is the constant sheaf on X. Next, define for any two presheaves $\mathcal{F}$ and $\mathcal{G}$ a presheaf

$$\mathcal{F} \otimes_{\mathrm{psh}} \mathcal{G} \colon U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U).$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $\mathcal{F} \otimes \mathcal{G} := (\mathcal{F} \otimes_{\mathrm{psh}} \mathcal{G})^{\mathrm{sh}}$. This gives a bifunctor

$$- \otimes - \colon \mathrm{Sh}(X, \mathbf{E}) \times \mathrm{Sh}(X, \mathbf{E}) \to \mathrm{Sh}(X, \mathbf{E}).$$

Lemma B.1.4 ([18, Lemma 2.15])**.** $(\mathrm{Sh}(X, \mathbf{E}), 1_X, \otimes)$ *is closed symmetric monoidal.*

We use the following fact without further reference: A monoid structure on an $\mathbf{E}$-sheaf $\mathcal{R}$ is equivalent to the data of a monoid structure on the sections of $\mathcal{R}$ such that the restriction maps $\mathcal{R}(U) \to \mathcal{R}(V)$ are multiplicative. Similarly, the structure which makes an $\mathbf{E}$-sheaf an $\mathcal{R}$-module object is equivalent to section-wise module structures which commute with the restriction maps in the obvious way.

Lemma B.1.5. $(\mathcal{F} \otimes_{\mathrm{psh}} \mathcal{G})^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{F}^{\mathrm{sh}} \otimes \mathcal{G}^{\mathrm{sh}}$ *is an isomorphism for any two* $\mathbf{E}$*-presheaves* $\mathcal{F}$ *and* $\mathcal{G}$ *on* X*. That is sheafification is strongly monoidal.*

Proof. See [18, Lemma 2.16]. □

We keep our assumptions on $\mathbf{E}$ fixed and consider a morphism $f\colon X \to Y$ of sites, which is given by a functor $f^{-1}\colon Y \to X$ between the underlying categories.

$$f^{\mathrm{psh},-1}\colon \operatorname{Psh}(Y,\mathbf{E}) \to \operatorname{Psh}(X,\mathbf{E})$$
$$f^{\mathrm{psh},-1}(\mathcal{F})(U) := \varinjlim_{U \to f^{-1}(V)} \mathcal{F}(V)$$

is the *presheaf inverse image*. The *direct image functor* is

$$f_*\colon \operatorname{Psh}(X,\mathbf{E}) \to \operatorname{Psh}(Y,\mathbf{E})$$
$$f_*(\mathcal{F})(U) := \mathcal{F}\left(f^{-1}(U)\right),$$

as discussed in [18, section 2.6]. f_* sends sheaves to sheaves but $f^{\mathrm{psh},-1}$ does, in general, not. This is why we define the direct image functor $f^{-1} := \cdot^{\mathrm{sh}} \circ f^{\mathrm{psh},-1}$.

Lemma B.1.6. *There is a natural isomorphism $f^{-1}(\mathcal{F} \otimes \mathcal{G}) \cong f^{-1}\mathcal{F} \otimes f^{-1}\mathcal{G}$ for any two $\mathbf{E}$-sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. That is, f^{-1} is strongly monoidal.*

Proof. See [18, Lemma 2.28]. □

Lemma B.1.7. *Sheafification commutes with restriction.*

Proof. This follows from the adjunctions, in particular $f^{-1} \dashv f_*$. □

Lemma B.1.7 implies the following.

Lemma B.1.8. *For any two $\mathbf{E}$-presheaves $\mathcal{F}$ and $\mathcal{G}$ on X, there is a functorial isomorphism $\mathcal{F}|_U \otimes \mathcal{G}|_U \cong (\mathcal{F} \otimes \mathcal{G})|_U$ for any $U \in X$.*

B.2 Sites with many quasi-compact open subsets

In this subsection, we fix again an elementary quasi-abelian category $\mathbf{E}$ and a site X.

Proposition B.2.1. *Suppose that any $U \in X$ is quasi-compact. Let $\mathcal{F}$ denote an $\mathbf{E}$-presheaf on X such that the sequence*

$$0 \to \mathcal{F}(U) \to \prod_{V \in \mathfrak{U}} \mathcal{F}(V) \to \prod_{W,W' \in \mathfrak{U}} \mathcal{F}(W \times_U W')$$

is strictly exact for every finite covering $\mathfrak{U}$ of any $U \in X$. Then $\mathcal{F}$ is a sheaf.

By [52, Corollary 1.2.28], Proposition B.2.1 follows from the following.

Lemma B.2.2. *Proposition B.2.1 holds for any elementary abelian category $\mathbf{E} = \mathbf{A}$.*

Proof. Let $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ denote a covering. Since U is quasicompact, we find a finite subcovering $\widetilde{\mathfrak{U}} = \{U_{\widetilde{i}} \to U\}_{\widetilde{i} \in \widetilde{I}}$ where $\widetilde{I} \subseteq I$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(U) & \xrightarrow{\alpha} & \prod_{i \in I} \mathcal{F}(U_i) & \xrightarrow{\beta} & \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j) \\ & & \| & & \downarrow{\scriptstyle \pi_{\widetilde{I}}} & & \downarrow{\scriptstyle \pi_{\widetilde{I} \times \widetilde{I}}} \\ 0 & \longrightarrow & \mathcal{F}(U) & \xrightarrow{\widetilde{\alpha}} & \prod_{\widetilde{i} \in \widetilde{I}} \mathcal{F}(U_{\widetilde{i}}) & \xrightarrow{\widetilde{\beta}} & \prod_{\widetilde{i},\widetilde{j} \in \widetilde{I}} \mathcal{F}(U_{\widetilde{i}} \times_U U_{\widetilde{j}}) \end{array} \tag{B.2.1}$$

where both $\pi_{\widetilde{I}}$ and $\pi_{\widetilde{I} \times \widetilde{I}}$ are the projections. It follows from the commutativity of the left square that α is a monomorphism. It remains to show that the canonical morphism $\Psi\colon \operatorname{im} \alpha \to \ker \beta$ is an isomorphism. We introduce some notation.

- Let $\widetilde{\Phi}$ denote the inverse of $\widetilde{\Psi}\colon \operatorname{im} \widetilde{\alpha} \to \ker \widetilde{\beta}$.
- π_{i_0} is the projection $\prod_{i \in I} \mathcal{F}(U_i) \to \mathcal{F}(U_{i_0})$ and $\alpha_{i_0} := \pi_{i_0} \circ \alpha$ for every $i_0 \in I$.
- $\alpha'^{-1}\colon \operatorname{im} \alpha \to \mathcal{F}(U)$ is the inverse of the morphism $\alpha'\colon \mathcal{F}(U) \to \operatorname{im} \alpha$ induced by α. It is an isomorphism because α is a monomorphism and $\mathbf{A}$ is abelian. Similarly, $\widetilde{\alpha}'^{-1}\colon \operatorname{im} \widetilde{\alpha} \to \mathcal{F}(U)$ denotes the inverse of the morphism $\widetilde{\alpha}'\colon \mathcal{F}(U) \to \operatorname{im} \widetilde{\alpha}$ induced by $\widetilde{\alpha}$.
- $\pi_{\widetilde{I}}$ restricts to maps $\pi_{\widetilde{I}}^{\operatorname{im} \alpha}\colon \operatorname{im} \alpha \to \operatorname{im} \widetilde{\alpha}$ and $\pi_{\widetilde{I}}^{\ker \beta}\colon \ker \beta \to \ker \widetilde{\beta}$.
- Write $\pi_{i_0}^{\operatorname{im} \alpha}$ for the composition $\operatorname{im} \alpha \hookrightarrow \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{\pi_{i_0}} \mathcal{F}(U_{i_0})$ and $\pi_{i_0}^{\ker \beta}$ for the composition $\ker \beta \hookrightarrow \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{\pi_{i_0}} \mathcal{F}(U_{i_0})$.

Lemma B.2.3. *We have identities*

$$\begin{aligned} \pi_{i_0}^{\operatorname{im} \alpha} &= \alpha_{i_0} \circ \widetilde{\alpha}'^{-1} \circ \pi_{\widetilde{I}}^{\operatorname{im} \alpha}, \quad \textit{and} \\ \pi_{i_0}^{\ker \beta} &= \alpha_{i_0} \circ \widetilde{\alpha}'^{-1} \circ \widetilde{\Phi} \circ \pi_{\widetilde{I}}^{\ker \beta} \end{aligned}$$

for every $i_0 \in I$.

Proof. Compose $\widetilde{\alpha}' = \pi_{\widetilde{I}}^{\operatorname{im} \alpha} \circ \alpha'$ with $\widetilde{\alpha}'^{-1}$ on the left and α'^{-1} on the right to get

$$\alpha'^{-1} = \widetilde{\alpha}'^{-1} \circ \pi_{\widetilde{I}}^{\operatorname{im} \alpha}.$$

Now we compose with α' on the left and again with $\pi_{i_0}^{\operatorname{im} \alpha}$ on the left. This yields

$$\begin{aligned} \pi_{i_0}^{\operatorname{im} \alpha} &= \pi_{i_0}^{\operatorname{im} \alpha} \circ \left(\alpha' \circ \widetilde{\alpha}'^{-1} \circ \pi_{\widetilde{I}}^{\operatorname{im} \alpha}\right) \\ &= \left(\pi_{i_0}^{\operatorname{im} \alpha} \circ \alpha'\right) \circ \widetilde{\alpha}'^{-1} \circ \pi_{\widetilde{I}}^{\operatorname{im} \alpha} \\ &= \alpha_{i_0} \circ \widetilde{\alpha}'^{-1} \circ \pi_{\widetilde{I}}^{\operatorname{im} \alpha} \end{aligned}$$

which is the first identity stated above in Lemma B.2.3.

To prove the second identity, we may assume without loss of generality that **A** is small; otherwise we pass to a suitable subcategory. The Freyd-Mitchell Embedding Theorem [60, Theorem 1.6.1] gives that **A** is the category of modules over some ring. Let $(s_i)_{i\in I} \in \ker\beta$. Then $\left(\widetilde{\Phi}\circ\pi_{\widetilde{I}}^{\ker\beta}\right)\left((s_i)_{i\in I}\right) = (s_{\widetilde{i}})_{\widetilde{i}\in\widetilde{I}}$ lies in the kernel of $\widetilde{\beta}$, thus it lies in the image of $\widetilde{\alpha}$. That is, there exists an $s \in \mathcal{F}(U)$ such that $\widetilde{\alpha}(s) = (s_{\widetilde{i}})_{\widetilde{i}\in\widetilde{I}}$. With other words, $s = \left(\widetilde{\alpha}'^{-1}\circ\widetilde{\Phi}\circ\pi_{\widetilde{I}}^{\ker\beta}\right)\left((s_i)_{i\in I}\right)$. We have to show that $\alpha_{i_0}(s) = s_{i_0}$ for all $i_0 \in I$, since $\pi_{i_0}^{\ker\beta}\left((s_i)_{i\in I}\right) = s_{i_0}$.

Because $\widetilde{\mathfrak{U}} = \{U_{\widetilde{i}} \to U\}_{\widetilde{i}\in\widetilde{I}}$ is a finite covering of U, $\{U_{\widetilde{i}}\times_U U_{i_0} \to U_{i_0}\}_{\widetilde{i}\in\widetilde{I}}$ is a finite covering of U_{i_0}. Thus the sequence

$$0 \longrightarrow \mathcal{F}(U_{i_0}) \longrightarrow \prod_{\widetilde{i}\in\widetilde{I}}\mathcal{F}(U_{\widetilde{i}}\times_U U_{i_0}) \longrightarrow \prod_{\widetilde{i},\widetilde{j}\in\widetilde{I}}\mathcal{F}((U_{\widetilde{i}}\times_U U_{i_0})\times_{U_{i_0}}(U_{\widetilde{j}}\times_U U_{i_0}))$$

is exact. That is, the element s_{i_0} is uniquely determined by its restrictions to the open sets $U_{\widetilde{i}}\times_U U_{i_0}$. Therefore, the computation

$$s_{i_0}|_{U_{\widetilde{i}}\times_U U_{i_0}} = s_{\widetilde{i}}|_{U_{\widetilde{i}}\times_U U_{i_0}} = \left(s|_{U_{\widetilde{i}}}\right)|_{U_{\widetilde{i}}\times_U U_{i_0}} = s|_{U_{\widetilde{i}}\times_U U_{i_0}}$$

implies $\alpha_{i_0}(s) = s|_{U_{i_0}} = s_{i_0}$. Note that the first equality $s_{i_0}|_{U_{\widetilde{i}}\times_U U_{i_0}} = s_{\widetilde{i}}|_{U_{\widetilde{i}}\times_U U_{i_0}}$ follows from $(s_i)_{i\in I} \in \ker\beta$. This finishes the proof. □

We claim that

$$\Phi = \alpha'\circ\widetilde{\alpha}'^{-1}\circ\widetilde{\Phi}\circ\pi_{\widetilde{I}}^{\ker\beta}$$

is a two-sided inverse of Ψ. First, we will show that it is a right-inverse. Since $\ker\beta$ is a subobject of $\prod_{i\in I}\mathcal{F}(U)$, it suffices to show the commutativity of the diagrams

$$\begin{array}{ccc} \ker\beta & \xrightarrow{\Psi\circ\Phi} & \ker\beta \\ & \searrow^{\pi_{i_0}^{\ker\beta}} & \downarrow{\scriptstyle \pi_{i_0}^{\ker\beta}} \\ & & \mathcal{F}(U_{i_0}) \end{array}$$

for every $i_0 \in I$. Now compute

$$\pi_{i_0}^{\ker\beta}\circ\Psi\circ\alpha' = \pi_{i_0}^{\operatorname{im}\alpha}\circ\alpha' = \alpha_{i_0} \tag{B.2.2}$$

and therefore

$$\begin{aligned} \pi_{i_0}^{\ker\beta}\circ(\Psi\circ\Phi) &= \pi_{i_0}^{\ker\beta}\circ\Psi\circ\alpha'\circ\widetilde{\alpha}'^{-1}\circ\widetilde{\Phi}\circ\pi_{\widetilde{I}}^{\ker\beta} \\ &\overset{(B.2.2)}{=} \alpha_{i_0}\circ\widetilde{\alpha}'^{-1}\circ\widetilde{\Phi}\circ\pi_{\widetilde{I}}^{\ker\beta} \\ &\overset{B.2.3}{=} \pi_{i_0}^{\ker\beta}. \end{aligned}$$

We find that Φ is a right-inverse of Ψ. It remains to compute that the diagrams

$$\begin{array}{ccc} \operatorname{im}\alpha & \xrightarrow{\Phi\circ\Psi} & \operatorname{im}\alpha \\ & \searrow^{\pi_{i_0}^{\operatorname{im}\alpha}} & \downarrow \pi_{i_0}^{\operatorname{im}\alpha} \\ & & \mathcal{F}(U_{i_0}) \end{array}$$

are commutative for every $i_0 \in I$. We have $\pi_{\widetilde{I}}^{\ker\beta} \circ \Psi = \widetilde{\Psi} \circ \pi_{\widetilde{I}}^{\operatorname{im}\alpha}$, so that composing with $\widetilde{\Phi}$ on the left yields

$$\widetilde{\Phi} \circ \pi_{\widetilde{I}}^{\ker\beta} \circ \Psi = \pi_{\widetilde{I}}^{\operatorname{im}\alpha}. \tag{B.2.3}$$

We get

$$\begin{aligned} \pi_{i_0}^{\operatorname{im}\alpha} \circ (\Phi \circ \Psi) &= \pi_{i_0}^{\operatorname{im}\alpha} \circ \alpha' \circ \widetilde{\alpha}'^{-1} \circ \widetilde{\Phi} \circ \pi_{\widetilde{I}}^{\ker\beta} \circ \Psi \\ &= \alpha_{i_0} \circ \widetilde{\alpha}'^{-1} \circ \widetilde{\Phi} \circ \pi_{\widetilde{I}}^{\ker\beta} \circ \Psi \\ &\overset{(\text{B.2.3})}{=} \alpha_{i_0} \circ \widetilde{\alpha}'^{-1} \circ \pi_{I}^{\operatorname{im}\alpha} \\ &\overset{\text{B.2.3}}{=} \pi_{i_0}^{\operatorname{im}\alpha}. \end{aligned}$$

That is, Φ is a left-inverse of Ψ as well. We have thus shown that the top row of the diagram (B.2.1) is exact. Since this choice of covering was arbitrary, $\mathcal{F}$ is a sheaf. □

Appendix C

Completeness of rings of formal power series

We prove the following result.

Proposition C.0.1. *Fix a regular sequence $s_1, \ldots, s_n$ in a commutative ring S. We consider the ideal $I := (s_1, \ldots, s_n)$ and pick an arbitrary $d \in \mathbb{N}_{\geq 1}$. If S is I-adically complete, then $S[\![X_1, \ldots, X_d]\!]$ is $(X_1, \ldots, X_d, s_1, \ldots, s_n)$-adically complete.*

Lemma C.0.2. *Let S denote a commutative ring, fix a regular sequence $s_1, \ldots, s_n \in S$, and consider the ideal $I := (s_1, \ldots, s_n)$. The following are equivalent.*

(i) S is I-adically complete.

(ii) S is, for every $i = 1, \ldots, n$, s_i-adically complete.

Proof. The direction (i) $\Rightarrow$ (ii) is proven in [59, Tag 090T]. Now suppose (ii). [59, Tag 091T] implies that S is, for every $i = 1, \ldots, n$, derived s_i-adically complete. It is derived I-adically complete by [59, Tag 091Q]. Now use the notation from [59, Tag 0BKC], that is consider the Koszul complexes $K_j^\bullet = K_\bullet\left(S; s_1^j, \ldots, s_n^j\right)$ for every $j \in \mathbb{N}_{\geq 1}$. Since $s_1, \ldots, s_n$ is a regular sequence, the sequences $s_1^j, \ldots, s_n^j$ are regular. [59, Tag 062F] gives that $K_j^\bullet$ is quasi-isomorphic to $S/\left(s_1^j, \ldots, s_n^j\right)$, viewed as a complex concentrated in degree zero. [59, Tag 091Z] implies that the canonical

$$S \xrightarrow{\cong} R\lim\left(S \otimes_S^{\mathrm{L}} K_j^\bullet\right) \cong R\lim S/\left(s_1^j, \ldots, s_n^j\right)$$

is an isomorphism. Here, the operator lim denotes the homotopy limit along the maps $\cdots \to K_2^\bullet \to K_1^\bullet$, see the discussion in [59, Tag 0BKC]. We apply [59, Tag 0941] [1] to see that the right-hand side coincides with $R\varprojlim_{t\in\mathbb{N}} S/\left(s_1^j, \ldots, s_n^j\right)$, where

[1] In the notation of [59, Tag 0941], we choose C to be the site associated to the topological space with one point.

$R\varprojlim_{j\in\mathbb{N}}$ denotes the derived functor of the inverse limit. In particular, taking zeroth cohomology, $S \cong \varprojlim_{j\in\mathbb{N}} S/\left(s_1^j,\dots,s_n^j\right)$. It suffices to show that the canonical morphism $\varprojlim_{j\in\mathbb{N}} S/\left(s_1^j,\dots,s_n^j\right) \to \varprojlim_{j\in\mathbb{N}} S/I^j$ is an isomorphism. But for every $j \geq n$, $I^j \subseteq \left(s_1^l,\dots,s_n^l\right)$ for $l = \lfloor j/n \rfloor$, proving the claim. $\square$

Lemma C.0.3. *Let S denote a commutative ring containing an element $s \in S$ such that it is s-adically complete. Then $S[\![X_1,\dots,X_d]\!]$ is s-adically complete as well.*

Proof. We have to show that the following morphism is an isomorphism:

$$\varphi\colon S[\![X_1,\dots,X_d]\!] \to \varprojlim_j S[\![X_1,\dots,X_d]\!]/s^j.$$

Let $f \in \ker\varphi$. That is for every j exists an $f_j \in S[\![X_1,\dots,X_d]\!]$ such that $f = s^j f_j$. Writing $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha X^\alpha$ and $f_j = \sum_{\alpha\in\mathbb{N}^d} f_{j\alpha} X^\alpha$ for all j, this means that $f_\alpha = s^j f_{j\alpha}$ for all α and for all j. Since S is s-adically separated this implies $f_\alpha = 0$ for all α. This shows $f = 0$. To prove surjectivity, pick $f_j \in S[\![X_1,\dots,X_d]\!]$ for all j such that

$$\left(\overline{f_j}\right)_j \in \varprojlim_j S[\![X_1,\dots,X_d]\!]/s^j,$$

where $\overline{f_j}$ denotes the image of f_j modulo s^j. This means that $f_l - f_j \in (s^j)$ for every $l \geq j$. Writing $f_j = \sum_{\alpha\in\mathbb{N}^d} f_{j\alpha} X^\alpha$ for all j, this implies that $f_{l\alpha} - f_{t\alpha} \in (s^j)$ for every $l \geq j$. Therefore

$$\left(\overline{f_{j\alpha}}\right) \in \varprojlim_j S/s^j.$$

Because S is s-adically complete, we conclude that there exists an $f_\alpha \in S$ for every α such that $f_\alpha \equiv f_{j\alpha} \mod s^j$. Set $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha X^\alpha$. We claim that $\varphi(f) = \left(\overline{f_j}\right)_j$. Indeed, we have

$$f - f_j = \sum_{\alpha\in\mathbb{N}^d} (f_\alpha - f_{j\alpha}) X^\alpha \equiv 0 \mod s^j$$

for every j. This proves Lemma C.0.3. $\square$

Proof of Proposition C.0.1. $S[\![X_1,\dots,X_d]\!] \cong S[\![X_1,\dots,X_{i-1},X_{i+1},\dots X_d]\!][\![X_i]\!]$ is X_i-adically complete for every $i = 1,\dots,d$. By Lemma C.0.2, S is s_i-adically complete for every $i = 1,\dots,n$, therefore $S[\![X_1,\dots,X_d]\!]$ is s_i-adically complete for every $i = 1,\dots,n$, see Lemma C.0.3. But the sequence $X_1,\dots,X_d,s_1,\dots,s_n$ is regular, thus we can apply Lemma C.0.2 to finish the proof of Proposition C.0.1. $\square$

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