TOWARDS A RIEMANN-HILBERT CORRESPONDENCE FOR $\widehat{\mathcal{D}}$-MODULES

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Contents

1. Background

1.1. Locally analytic representations and the Arens-Michael envelope. Throughout this text, K denotes a field complete with respect to a non-trivial non-Archimedean valuation.

Also, let L be a finite extension of $\mathbb{Q}_p$ and assume that the ground field K contains L. Pick a locally L-analytic group G. The theory of *admissible locally analytic G-representations* in locally convex K-vector spaces is due to Schneider and Teitelbaum [1] [2] [3] [4], and it admits wide-ranging applications, for example in the p-adic local Langlands program [5] [6] [7].

In order to better understand this class of representations, Ardakov-Wadsley [8] [9] developed their theory of *$\widehat{\mathcal{D}}$-modules on rigid analytic spaces*, in order to replicate the classical *Beilinson-Bernstein localisation* in this setting. Let $\mathfrak{g}$ denote a complex semi-simple Lie algebra of an algebraic group, and let $\mathcal{B}$ denote its flag variety. Recall that in their article [10], Beilinson and Bernstein established an equivalence between the category of finitely generated modules over the enveloping algebra $U(\mathfrak{g})$ with a fixed regular infinitesimal central character χ and the category of coherent modules for the sheaf of χ-twisted differential operators on $\mathcal{B}$. Therefore, one can study representations of $\mathfrak{g}$ via twisted $\mathcal{D}$-modules on $\mathcal{B}$.

In what follows, we give a brief summary of Ardakov-Wadsley's work, closely following [11]. We first note that their analogue of the Beilinson-Bernstein localisation theorem [8, Theorem E] does not directly apply to locally analytic representations. The class of representations that they consider is a certain category of modules over the *Arens-Michael envelope* of a finite dimensional Lie algebra over K, in analogy to the enveloping algebra of the complex semi-simple Lie algebra $\mathfrak{g}$ in the classical theory.

Definition 1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over K.*

(i) *A* Lie lattice *in $\mathfrak{g}$ is a finitely generated K°-submodule L of $\mathfrak{g}$ which satisfies $[L, L] \subseteq L$ and which spans $\mathfrak{g}$ as a K-vector space.*

(ii) *The* affinoid enveloping algebra *of a Lie lattice L in $\mathfrak{g}$ is*

$$\widehat{U(L)_K} := \left(\varprojlim U(L)/(\pi^\alpha)\right) \otimes K.$$

(iii) *The* Arens-Michael envelope *of $U(\mathfrak{g})$ is*

$$\widehat{U(\mathfrak{g})} := \varprojlim \widehat{U(L)_K},$$

where the inverse limit is taken over all possible Lie lattices.

In the classical Beilinson-Bernstein localisation, one considers finitely generated modules over the enveloping algebra of the complex semi-simple Lie algebra of interest. We aim to get a similar finiteness condition on the category of modules of the Arens-Michael envelope of $U(\mathfrak{g})$. It is not a reasonable idea to consider finitely generated modules, because Arens-Michael envelopes are non-noetherian rings whenever $\mathfrak{g}$ is non-zero. In particular, the category of finitely generated modules over $\widehat{U(\mathfrak{g})}$ will not be abelian. To get around this, Schneider and Teitelbaum introduced the following finiteness condition, which gives the correct analogue of "finitely generated" in this setting.

Definition 2. (i) *A* Fréchet-Stein algebra *is the projective limits of a diagram $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \dots$ of Noetherian K-Banach algebras such that*

- *A_{n+1} has dense image in A_n for all $n \geq 0$, and*
- *A_n is a flat right A_{n+1}-module for all $n \geq 0$.*

(ii) *A left A-module M is* co-admissible *if $A_n \otimes_A M$ is a finitely generated A_n-module for all $n \geq 0$, and the natural map $M \to \varprojlim A_n \otimes_A M$ is a bijection.*

(iii) *$\mathcal{C}_A$ is the full subcategory of left A-modules consisting of the co-admissible A-modules.*

Schneider and Teitelbaum proved that $\mathcal{C}_A$ is always an abelian category whenever A is Fréchet-Stein, and they showed that Arens-Michael envelopes are Fréchet-Stein. This gives us a well-behaved category of modules over $\widehat{U(\mathfrak{g})}$, and thus one half of Ardakov-Wadsley's Beilinson-Bernstein correspondence.

Before we start our discussion of the second half of their correspondence, we would like to point out how the Arens-Michael envelope is related to locally analytic representations. By definition, the *locally analytic distribution algebra of G over K* is the strong dual $D(G,K)$ of the vector space of locally analytic K-valued functions on G. When the group G is compact, Schneider and Teitelbaum showed that $D(G,K)$ is a Fréchet-Stein algebra, and thus there is an abelian category of coadmissible $D(G,K)$-modules. We say that a locally analytic representation V of an arbitrary locally L-analytic group G is *admissible* if its strong dual is coadmissible as a module over the distribution algebra $D(H,K)$ of every compact open subgroup H of G.

There is a natural embedding of the Lie algebra $\mathfrak{g}$ of G into $D(G,K)$, which extends to an embedding of K-algebras $U(\mathfrak{g}_K) \hookrightarrow D(G,K)$, where $\mathfrak{g} := K \otimes_L \mathfrak{g}$. It follows from the work of Kohlhaase [12] that the closure of the image is the Arens-Michael envelope of $U(\mathfrak{g}_K)$. In particular, we approximate admissible locally analytic G-representations via our study of the Arens-Michael envelope. However, we would like to mention that there is a Beilinson-Bernstein localisation due to Ardakov [13], which relates admissible locally analytic representations of semisimple p-adic Lie groups to some category of coadmissible *equivariant* $\mathcal{D}$-modules on smooth rigid analytic varieties.

1.2. $\widehat{\mathcal{D}}$-modules on rigid analytic spaces.

We are now going to discuss the other half of Ardakov-Wadsley's Beilinson-Bernstein correspondence, which is analogous to the $\mathcal{D}$-modules on the flag variety in the classical result by Beilinson and Bernstein.

Suppose that $\mathfrak{g}$ is a split semisimple Lie algebra over K. We note that any Lie lattice L on an affinoid K-variety X can be viewed as a *Lie-Reinhart algebra*, see [14], over $(K^\circ, \mathcal{O}(X)^\circ)$, and such has an enveloping algebra $U(L)$. Therefore, the following definition makes sense.

Definition 3. *Let X be an affinoid K-variety, and let $\mathcal{T}(X) := \mathrm{Der}_K\, \mathcal{O}(X)$.*

(a) *A* Lie lattice *on X is any finitely generated $\mathcal{O}(X)^\circ$-submodule L of $\mathcal{T}(X)$ such that $[L,L] \subseteq L$ and L spans $\mathcal{T}(X)$ as a K-vector space.*

(b) *For any Lie lattice L on X we have the Noetherian Banach algebra*

$$\widehat{U(L)_K} := \left(\varprojlim U(L)/(\pi^a)\right) \otimes_R K$$

(c) $\widehat{\mathcal{D}} := \varprojlim \widehat{U(L)_K}$*, the inverse limit being taking over all possible Lie lattices L in $\mathcal{T}(X)$.*

It turns out that the previous Definition 3 gives rise to a sheaf.

Theorem 4 ([8])**.** *Let X be a smooth rigid analytic space.*

(i) *$\widehat{\mathcal{D}}$ extends to a sheaf of K-Fréchet algebras on X.*

(ii) *If X is affinoid and $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, then $\widehat{\mathcal{D}}(X)$ is a Fréchet-Stein algebra.*

Statement (ii) of Theorem 4 enables us to introduce a notion of coadmissibility of $\widehat{\mathcal{D}}$-modules.

Definition 5. *Let X be smooth rigid analytic space. A sheaf of $\widehat{\mathcal{D}}$-modules $\mathcal{M}$ on X is* co-admissible *if there is an admissible covering $\{X_i\}$ of X such that $\mathcal{T}(X_i)$ is a free $\mathcal{O}(X_i)$-module, and $\mathcal{M}(X_i) \in \mathcal{C}_{\widehat{\mathcal{D}}(X_i)}$ for all i. We denote the category of all co-admissible $\widehat{\mathcal{D}}$-modules on X by $\mathcal{C}_X$.*

We have now gathered all the terminology in order to state the Beilinson-Bernstein equivalence.

Theorem 6 ([8, Theorem E])**.** *Let $\mathbb{G}$ be a connected split reductive group over K with Lie algebra $\mathfrak{g}$, let $\mathcal{B}^{an}$ be the rigid analytic flag variety and let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$. Then there is an equivalence of abelian categories*

$$\left\{ \begin{array}{l} \textit{co-admissible} \\ \widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g})} K\textit{-modules} \end{array} \right\} \simeq \left\{ \begin{array}{l} \textit{co-admissible sheaves of} \\ \widehat{\mathcal{D}}\textit{-modules on } \mathcal{B}^{\mathrm{an}} \end{array} \right\}$$

1.3. **The failure of holonomicity.** The classical Riemann-Hilbert correpondence is one of the most important theorems in the theory of $\mathcal{D}$-modules. It builds a bridge between analysis and toplogy, and leads to many applications in various fields.

Theorem 7. *Let X be a smooth complex algebraic variety. Then the de Rham functor is an equivalene of categories*

$$\mathrm{DR}\colon \mathbf{D}^b_{rh}(\mathcal{D}_X) \to \mathbf{D}^b_c(\mathbb{C}_{X^{\mathrm{an}}}).$$

It sends regular holonomic $\mathcal{D}_X$-modules to perverse sheaves on X.

Recall from the previous subsection that the Beilinson-Bernstein localisation enables us to view certain $\mathfrak{g}$-modules, where $\mathfrak{g}$ denotes a complex semi-simple Lie algebra, as $\mathcal{D}$-modules on a certain flag variety. Now the Riemann-Hilbert correspondence enables us to reformulate these $\mathfrak{g}$-module as perverse sheaves on this flag variety. These techniques has significant applications, for instance the proof of the Kazhdan-Lusztig conjectures [15] given by [16].

Inspired by the far reaching applications of the classical Riemann-Hilbert correspondence in the complex setting, the author aims to develop an anologue of the classical Riemann-Hilbert correspondence for $\widehat{\mathcal{D}}$-modules on rigid analytic spaces. And as Ardakov mentions in [11], "there are some mildly encouraging signs that *some* such analogue [of the classical Riemann-Hilbert correspondence] exists".

However, there are still many difficulties to overcome. For example, there does not exist a good theory of characteristic varieties (yet). This is a major issue, since the definition of Holonomicity relies heavily on this notion: Recall [17, Section 3.1] that if X denotes a smooth complex algebraic variety, then a coherent $\mathcal{D}$-module M is *holonomic* if the dimension of the characteristic variety of M coincides with the dimension of X, or $M = 0$. Nevertheless, Ardakov-Bode-Wadsley [18] made the following

Definition 8. *Let X be a smooth affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, and let M be a co-admissible $D := \widehat{\mathcal{D}}(X)$-module.*

(i) *The* grade *of M is $j(M) = \min\left\{j \in \mathbb{N}\colon \operatorname{Ext}^j_D(M, D) \neq 0\right\}$,*
(ii) *the* dimension *of M is $d(M) := 2\dim X - j(M)$, and*
(iii) *M is* weakly holonomic *if $d(M) = \dim X$.*

Furthermore, recall that one important ingredient in the proof of Theorem 7 is that the class of holonomic $\mathcal{D}$-modules is preserved by push-forwards f_+, pull-backs f^+, and duality $\mathbb{D}$. For example, one can check that if $\iota\colon Y \hookrightarrow X$ is a closed embedding of smooth rigid analytic spaces, then ι_+ preserves weaky holonomic $\widehat{\mathcal{D}}$-modules. However, ι^+ does *not* preserve weak holonomicity! We refer to [18] for more details. In particular, the category of weakly holonomic $\widehat{\mathcal{D}}$-modules seems to be too large to be suitable for a Riemann-Hilbert-correspondence for $\widehat{\mathcal{D}}$-modules. This motivates to look somewhere else for a good theory of Holonomicity.

Whilst the author was writing this text, Andreas Bode made his preprint [19] available, in which he was able to develop a six-functor formalism for $\widehat{\mathcal{D}}$-modules. In particular, he considers the category $\widehat{\mathcal{B}c}_K\left(\widehat{\mathcal{D}}\right)$ of *bornological* $\widehat{\mathcal{D}}$-modules. This has two critical advantages.

(i) $\widehat{\mathcal{B}c}_K\left(\widehat{\mathcal{D}}\right)$ retains the analytic flavour of the theory, and
(ii) it has a well-behaved derived category $\mathbf{D}(\widehat{\mathcal{D}})$ taking all the analytic nature into account.

Bode then defines all necessary six functors $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_X}-, \mathbb{D}, f_+, f^+, f_!$, and $f^!$, as in the classical $\mathcal{D}$-module theory. These functors turn out to be well-behaved, when X is a smooth rigid analytic K-variety and one restricts to a certain triangulated subcategory of $\mathbf{D}(\widehat{\mathcal{D}})$, namely the category of $\mathcal{C}$*-complexes*.

Having these six-functors at hand, one could now start thinking about a suitable category of bornological $\widehat{\mathcal{D}}_X$-modules that they preserve. This could give rise to a category of *holonomic* $\widehat{\mathcal{D}}$-modules.

However, the author of this article pursues a different approach to develop an analogue of the Riemann-Hilbert correspondence, which we will discuss in the following subsections. Because of the same reasons mentioned above, this approach heavily relies on the theory of bornological K-vector spaces and sheaves with values in the category $\mathbf{CBorn}_K$ of complete convex bornological K-vector spaces. Thus we start by giving an introduction to this subject in the following subsection.

1.4. **Bornological K-vector spaces.** In this subsection, we recall the definition of bornological K-vector spaces, with a particular focus on the symmetric monoidal structure on the category $\mathbf{CBorn}_K$ of complete convex bornological K-vector spaces. Our main references are [20] and [21]. We note that the first of these two references only address the case $K = \mathbb{C}$, but the relevant proofs carry over to our non-Archimedean setting without change.

Definition 9. *Let E be a set. A* bornology *on E is a collection $\mathcal{B}$ of subsets of E such that*

(i) *$\mathcal{B}$ is a covering of E, that is for every $e \in E$ exists an element $B \in \mathcal{B}$ such that $e \in \mathcal{B}$,*
(ii) *$\mathcal{B}$ is stable under inclusions, that is any subset of a set $B \in \mathcal{B}$ is again contained in $\mathcal{B}$, and*
(iii) *$\mathcal{B}$ is stable under finite unions, that is for any finite collections of elments $B_1, \dots, B_n \in \mathcal{B}$, we have $\bigcup_{i=1}^n B_i \in \mathcal{B}$.*

The pair $(E, \mathcal{B})$ is a bornological set*, and the elements of $\mathcal{B}$ are* bounded subsets *of E (with respect to $\mathcal{B}$). A family of subsets $\mathcal{A} \subseteq \mathcal{B}$ is a* basis *for $\mathcal{B}$ if for any bounded set $B \in \mathcal{B}$ there exist $A_1, \dots, A_n \in \mathcal{A}$ such that $B \subseteq \bigcup_{i=1}^n A_i$. A map* *of bornological sets* $f\colon (E,\mathcal{B}_E) \to (F,\mathcal{B}_F)$ *is* bounded *if it sends bounded sets to bounded sets, that is* $f(B) \in \mathcal{B}_F$ *for all* $B \in \mathcal{B}_E$.

Definition 10. *A* bornological K-vector space *is a* K*-vector space* E *along with a bornology on the underlying set of* E *such that*

(i) $B_1 + B_2$ *is bounded if* $B_1, B_2 \subseteq E$ *are, and*
(ii) $D \cdot B$ *is bounded if* $D \subseteq K$ *and* $B \subseteq E$ *are.*

We will be particularly interested in bornological vector spaces whose bounded subsets can be described using convex subsets, in the following way. Let $K^\circ \subseteq K$ denote the subring of *power-bounded elements*. We say that a subset of a K-vector space E is *absolutely convex* if it is a K°-submodule of E.

Definition 11. *A bornological* K*-vector space is said to be* convex *or* of convex type *if it has a basis made of absolutely convex subsets. We will denote by* $\mathbf{Born}_K$ *the category whose objects are the bornological* K*-vector spaces of convex type and whose morphisms are bounded linear maps between them.*

We remark that $\mathbf{Born}_K$ is complete and cocomplete, and the limits and colimits have the following description.

Lemma 12. *Suppose we are given a functor* $F\colon S \to \mathbf{Born}_K$*, where* S *is a small category.*

(i) *Then the limit of* F *is given as follows: Its underlying vector space* V_F *is the limit of* F *in the category of* K*-vector spaces, and declare a subset of* V_F *to be bounded if its image in each* $F(s)$ *is bounded.*
(ii) *Then the colimit of* F *is given as follows: Its underlying vector space* V^F *is the colimit of* F *in the category of* K*-vector spaces, and declare a subset* $B \subseteq V^F$ *to be bounded, if there exists a bounded subset* $B_s \subseteq F(s)$ *such that*
$$B \subseteq r_s(B_s),$$
where r_s *denotes the natural linear map* $F(s) \to V_F$.

Proof. The first statement can be found in [21, Remark 3.45]. Also, the same reference says that the colimit of F is given as follows: Its underlying vector space V^F is the colimit of F in the category of K-vector spaces, and the vector space bornology $\mathcal{B}$ on V^F is generated by the images of bounded sets from the $F(s)$. Thus we have to show that this bornology coincides with the one described in the lemma, which we will denote by $\mathcal{B}'$.

First, it is clear that $\mathcal{B} \subseteq \mathcal{B}'$. Next, pick an arbitrary $B \in \mathcal{B}'$. By definition, there exists an $s \in S$ and a bounded subset $B_s \subseteq F(s)$ such that $B \subseteq r_s(B_s)$. Since $r_s(B_s) \in \mathcal{B}$, it follows that $B \in \mathcal{B}$ and therefore $B' \subseteq \mathcal{B}$. This shows $\mathcal{B} = \mathcal{B}'$. □

The category $\mathbf{CBorn}_K$ is *not* abelian. However, we have the following

Lemma 13. *The category* $\mathbf{Born}_K$ *is quasi-abelian.*

Proof. See [20, Proposition 1.8]. □

The notion of quasi-abelian categories as well as sheaves with values in quasi-abelian categories was developed by Schneiders [22]. The definition of a quasi-abelian category is almost similiar to the definition of an abelian category: It is an additive category with all kernels and cokernels, but we do not require every morphism in it to be strict. Recall that a morphism $f: E \to F$ in such a category is strict if the canonical morphism

$$\operatorname{coim} f \to \operatorname{im} f$$

is an isomorphism. This is the case in every abelian category, by definition. However, in a quasi-abelian category the class of strict morphism just satisfies two technical stability conditions, which we won't recall here.

We note that the main reason why we are interested in quasi-abelian categories is that they admit well-behaved derived categories. Thus they allow us to do homological algebra with categories as $\mathbf{Born}_K$. We will discuss this derived category in a later section of this text.

Another result of Schneiders is that the category of sheaves with values in a quasi-abelian category is well-behaved when this quasi-abelian category is *elementary*. Again, we won't recall the precise definition here, but we note that $\mathbf{Born}_K$ is *not* elementary. Luckily, it turns out that there exists a full subcategory of complete convex bornological K-vector spaces $\mathbf{CBorn}_K$ of $\mathbf{Born}_K$ that is elementary quasi-abelian, which we will now define.

First, let $\mathbf{Ban}_K$ denote the category of K-Banach spaces with continuous linear maps between them. By [23, Corollary 2.1.8.3], a linear map between K-Banach spaces is continuous if and only if it is bounded. We use this result troughout this text without reference. In particular, this gives a functor $\mathbf{Ban}_K \to \mathbf{Born}_K$, which makes it possible to view any diagram $I \to \mathbf{Ban}_K$ as a diagram $I \to \mathbf{Born}_K$

Definition 14. (i) *A bornological K-vector space is* separated *if its only bounded vector subspace is the trivial subspace $\{0\}$. We denote the full subcategory of $\mathbf{Born}_K$ of seperated convex bornological K-vector spaces by $\mathbf{SBorn}_K$.*

(ii) *A bornological space E over K is* complete *if there is a small filtered category I, a functor $I \to \mathbf{Ban}_K$, and an isomorphism*

$$E \simeq \varinjlim_{i \in I} E_i$$

for a filtered colimit of Banach spaces over K for which the system of morphisms are all injective. Note that this is a colimit in the category $\mathbf{Born}_K$. We denote the full subcategory of $\mathbf{Born}_K$ of complete convex bornological K-vector spaces by $\mathbf{CBorn}_K$.

With these definitions, we have fully faithful embeddings

$$\mathbf{CBorn}_K \hookrightarrow \mathbf{SBorn}_K \hookrightarrow \mathbf{Born}_K\,.$$

It turns out that both these inclusion admits left adjoints:

$$\operatorname{sep}\colon \mathbf{Born}_K \to \mathbf{SBorn}_K$$

is the *separation* and

$$\widehat{-}\colon \mathbf{SBorn}_K \to \mathbf{CBorn}_K$$

is the *completion*. Their composition is the *separated completion*, and we sometimes abbreviate $\widehat{E} = \widehat{\operatorname{sep}(E)}$ for $E \in \mathbf{Born}_K$. We refer to [21, Subsection 3.3] for the precise definition of these functors.

The following result is the reason for why we care about complete convex bornological K-vector spaces.

Lemma 15. *The category* $\mathbf{CBorn}_K$ *is elementary quasi-abelian with all limits and colimits and enough flat projectives.*

Proof. See [21, Lemma 3.46] □

The following lemma gives a description of the limits in colimits in $\mathbf{CBorn}_K$.

Lemma 16. *Suppose we are given a functor* $F: S \to \mathbf{CBorn}_K$*, where* S *is a small category.*

(i) *Then the limit of* F *is given as follows: Its underlying vector space* V_F *is the limit of* F *in the category of* K*-vector spaces, and declare a subset of* V_F *to be bounded if its image in each* $F(s)$ *is bounded.*
(ii) *Then the colimit of* F *is the seperated completion of the following bornological space: Its underlying vector space* V^F *is the colimit of* F *in the category of* K*-vector spaces, and the vector space bornology on* V^F *is generated by the images of bounded sets from the* $F(s)$*.*

Proof. The first statement can be found in [21, Remark 3.45]. Also, the same reference says that the colimit of F is given as the seperated completion of the colimit of F in $\mathbf{Born}_K$, which we have already described in Lemma 12. □

Comparing Lemma 12 and Lemma 16, we see that the small limits in $\mathbf{Born}_K$ and $\mathbf{CBorn}_K$ coincide. However, the small colimits in $\mathbf{CBorn}_K$ are the seperated completions of the small colimits in $\mathbf{Born}_K$, which makes them more difficult to handle. Fortunately, it is often possible to bypass this issue via the following

Lemma 17. *Let* $E: I \to \mathbf{Ban}_K$ *be a diagram for which the system of morphisms are all injective. Then we can calculate the colimit of* E *in* $\mathbf{CBorn}_K$ *in* $\mathbf{Born}_K$*. That is,*

$$\varinjlim_{i\in I}{}^{\mathbf{CBorn}_K} E_i = \varinjlim_{i\in I}{}^{\mathbf{Born}_K} E_i.$$

Proof. Recall that the colimit of E in $\mathbf{CBorn}_K$ is defined to be the completion of $\varinjlim_{i\in I}{}^{\mathbf{Born}_K} E_i$. But the colimit of E in $\mathbf{Born}_K$ is already complete, by definition. □

Let E be a complete convex bornological K-vector space with K-vector subspace F. The bornology on E gives rise to a bornology on F: The set

$$\{B \cap E : B \text{ bounded in } E\}$$

is the *subspace* or *induced bornology* on F. The same is true for quotients: If $q: E \to E/F$ denotes the canonical morphism, the set

$$\{q(B) : B \text{ bounded in } E\}$$

is the *quotient bornology* on E/F.

We can now describe monomorphisms, epimorphisms as well as strictness in $\mathbf{CBorn}_K$.

Lemma 18. *Let* $f: E \to F$ *be a morphism in* $\mathbf{CBorn}_K$*. Then*

(i) f *is a monomorphism if it is injective,*
(ii) f *is an epimorphism if* $f(E)$ *is bornologically dense in* F*,*
(iii) f *is a strict monomorphism if and only if it is injective, the bornology on* E *agrees with the induced bornology from* F *and* E *is a closed subspace of* F*, and*

(iv) *f is a strict epimorphism if and only if it is surjective and F is endowed with the quotient bornology.*
(v) *f is a strict if and only if $f(E)$ is closed and for any bounded subset B of F, there is a bounded subset B' of E such that $B \cap f(E) = f(B')$.*

Proof. We cite the first four results from [21, Proposition 3.60], so it remains to show (v). But this result follows from (iii) and (iv), as well as the following remark from [22, Remark 1.1.2]: f is strict if and only if $f = m \circ e$, where m is a strict monomorphism and e is a strict epimorphism. □

Next, we have the following characterisation of kernels, cokernels, images, and coimages in $\mathbf{CBorn}_K$.

Lemma 19. *Let $f \colon E \to F$ be a morphism in $\mathbf{CBorn}_K$. Then*
(i) $\ker(f) \simeq f^{-1}(0)$ *with the induced bornology of E.*
(ii) $\operatorname{coker}(f) \simeq F/\overline{f(E)}$ *with the induced bornology of F.*
(iii) $\operatorname{im}(f) \simeq \overline{f(E)}$ *with the quotient bornology.*
(iv) $\operatorname{coim}(f) \simeq E/\ker(f)$ *with the quotient bornology.*

Proof. See [20, Proposition 4.6]. □

Next, we discuss the closed symmetric monoidal structure on $\mathbf{CBorn}_K$. First, we introduce the internal homomorphism functor. Let E and F be bornological K-vector spaces. We define $\underline{\operatorname{Hom}}_K(E, F)$ to be the vector space $\operatorname{Hom}_K(E, F)$ of all bounded K-linear maps $E \to F$ equipped with the *equiboundedness bornology.* That is $\Phi \subseteq \underline{\operatorname{Hom}}_K(E, F)$ is bounded if for any bounded subset $B \subseteq E$, the set

$$\Phi(B) = \bigcup_{f \in \Phi} f(B)$$

is bounded in F.

Next, we define the tensor product. Again, let E and F be two convex bornological vector spaces. Then we define on $E \otimes_K F$ the *projective tensor product bornology* as follows. A basis for the bornology is given by absolutely convex hulls of subsets of the form

$$A \otimes B = \{x \otimes y \colon x \in A, y \in B\}$$

where A, B varies over a basis of absolutely convex subsets for the bornologies of E and F. Now we define

$$E \widehat{\otimes}_K F = \widehat{E \otimes_K F}$$

for any $E, F \in \mathbf{CBorn}_K$, the *completed projective tensor product.*

Theorem 20. *The previous constructions turn $\mathbf{CBorn}_K$ into a closed symetric monodial category. In particular, there exist natural isomorphisms*

$$\operatorname{Hom}_K(E \widehat{\otimes}_K F, G) \simeq \operatorname{Hom}_K(E, \underline{\operatorname{Hom}}_K(F, G)),$$

and

$$\underline{\operatorname{Hom}}_K(E \widehat{\otimes}_K F, G) \simeq \underline{\operatorname{Hom}}_K(E, \underline{\operatorname{Hom}}_K(F, G)),$$

for all complete convex bornological K-vector spaces E, F, G.

Proof. We cite that $\mathbf{CBorn}_K$ is a closed symmetric monoidal as well as the second natural isomorphism from [21, Remark 3.49]. Since the underlying vector space of $\underline{\operatorname{Hom}}_K$ is given by Hom_K, the first natural isomorphism follows from this. □

Recall that the category $\mathbf{Ban}_K$ also comes with a closed symmetric monoidal structure. For any two K-Banach spaces V and W, the underlying K-vector space of $\underline{\mathrm{Hom}}_K(V,W)$ is the K-vector space $\mathrm{Hom}_K(V,W)$ of bounded linear maps $f\colon V \to W$, equipped with the operator norm

$$\|f\| := \sup_{v\in V\setminus\{0\}} \frac{\|f(v)\|}{\|v\|}.$$

The monoidal structure is the usual completed tensor product. For details, we refer to to [24, Section A.3]. In the following Proposition 22, we compare the closed symmetric monoidal structures on $\mathbf{Ban}_K$ and $\mathbf{CBorn}_K$ with each other. We will use this result in the remainder of this article without future reference.

Lemma 21. *Let E be a K-Banach space, and $\pi \in K$ a uniformiser. Then for any $e \in E$, there exists a scalar $\lambda \in K^\times$ such that*

$$|\pi| < \|\lambda e\| \leq 1.$$

Proof. Set $\lambda := \pi^{-\lfloor \log_{|\pi|} \|e\| \rfloor + 1}$. If $|\pi^m| < \|e\| \leq |\pi^{m-1}|$, one computes $\lambda = \pi^{-m+1}$ and checks that $|\pi| < \|\lambda e\| \leq 1$. □

Proposition 22. *Denote the functor that sends a K-Banach space to its associated complete convex bornological K-vector space by*

$$(-)^b\colon \mathbf{Ban}_K \to \mathbf{CBorn}_K,$$

and pick two K-Banach spaces E and F.

(i) *We have a functorial isomorphism*

$$\underline{\mathrm{Hom}}_K\left(E^b, F^b\right) = \underline{\mathrm{Hom}}_K\left(E, F\right)^b$$

as well as

(ii) *a functorial isomorphism*

$$E^b \widehat{\otimes}_K F^b = \left(E \widehat{\otimes}_K F\right)^b.$$

Proof.

(i) Clearly, the underlying K-vector spaces of $\underline{\mathrm{Hom}}_K\left(E^b, F^b\right)$ and $\underline{\mathrm{Hom}}_K\left(E, F\right)^b$ coincide, so it is enough to compare their bornologies. In order to do so, pick an arbitrary bounded subset $\Phi \subseteq \underline{\mathrm{Hom}}_K\left(E^b, F^b\right)$. Define $B = \{e \in E\colon |\pi| < \|e\| \leq 1\}$. By definition, the set $\Phi(B)$ is bounded, say by some constant $C > 0$. Now pick arbitrary elements $f \in \Phi$ and $e \in E \setminus \{0\}$. By Lemma 21, there exists a scalar $\lambda \in K$ such that $\lambda e \in B$. We compute

$$\frac{\|f(e)\|}{\|e\|} = \frac{\|f(\lambda e)\|}{\|\lambda e\|} < |\pi|^{-1} C.$$

In particular, the set Φ is bounded with respect to the norm on $\underline{\mathrm{Hom}}_K\left(E, F\right)$, by the constant $|\pi|^{-1}C$.

To prove the converse, pick a bounded subset $\Phi \subseteq \underline{\mathrm{Hom}}_K\left(E, F\right)^b$, and assume it is bounded by some constant $C > 0$. That is, for every $f \in \Phi$ and for every $e \in E \setminus \{0\}$, we have

$$\|f(e)\| < C\|e\|.$$

Now pick an arbitrary bounded subset $B \subseteq E^b$, say by a constant $D > 0$. Then we have for every element $b \in B$

$$\|f(b)\| < C\|b\| < CD.$$

In particular, $\Phi(B)$ is bounded by CD. Since the choice of B was arbitrary, it follows that Φ is a bounded subset of $\underline{\mathrm{Hom}}_K\left(E^b, F^b\right)$.

(ii) Let G be an arbitrary complete convex bornological K-vector space, and write it as a colimit in $\mathbf{Born}_K$ $G = \varinjlim_{i\in I} G_i^b$. Now we apply (i) to compute

$$\begin{aligned}
\mathrm{Hom}_K\left(E^b\widehat{\otimes}_K F^b, G\right) &= \varinjlim_{i\in I} \mathrm{Hom}_K\left(E^b\widehat{\otimes}_K F^b, G_i^b\right)\\
&= \varinjlim_{i\in I} \mathrm{Hom}_K\left(E^b, \underline{\mathrm{Hom}}_K\left(F^b, G_i^b\right)\right)\\
&= \varinjlim_{i\in I} \mathrm{Hom}_K\left(E^b, \left(\underline{\mathrm{Hom}}_K\left(F, G_i\right)\right)^b\right)\\
&= \varinjlim_{i\in I} \mathrm{Hom}_K\left(E, \underline{\mathrm{Hom}}_K\left(F, G_i\right)\right)\\
&= \varinjlim_{i\in I} \mathrm{Hom}_K\left(E\widehat{\otimes}_K F, G_i\right)\\
&= \varinjlim_{i\in I} \mathrm{Hom}_K\left(\left(E\widehat{\otimes}_K F\right)^b, (G_i)^b\right)\\
&= \mathrm{Hom}_K\left(\left(E\widehat{\otimes}_K F\right)^b, G\right)
\end{aligned}$$

Finally, apply the Yoneda Lemma.

□

We have now gathered all the necessary information on the category of complete convex bornological K-vector spaces. As a next step, we now consider

1.5. **Sheaves with values in $\mathbf{CBorn}_K$.** Let X be a rigid analytic K-variety. We denote by X_w the category whose objects are the affinoid subdomains of X and whose morphisms are the inclusion. On this category, we consider the weak Grothendieck topology.

In this subsection, we consider the category of sheaves on X_w with values in $\mathbf{CBorn}_K$. This is a special case of Schneider's theory of sheaves with values in quasi-abelian categories [22].

Definition 23.

(i) *A* complete convex bornological presheaf on X *is a functor*

$$\mathcal{F}\colon X_w^{\mathrm{op}} \to \mathbf{CBorn}_K\,.$$

A morphism *between two such presheaves is a natural transformation. We denote the category of complete convex bornological presheaves on X by* $\mathrm{Psh}(X, \mathbf{CBorn}_K)$.

(ii) *A complete convex bornological presheaf $\mathcal{F}$ on X is a* complete convex bornological sheaf *if for any admissible open $U \in X_w$ and any admissible covering $\mathfrak{U}$ of U we get a strict exact sequence*

$$0 \to \mathcal{F}(U) \to \prod_{V\in\mathfrak{U}} \mathcal{F}(V) \to \prod_{W,W'\in\mathfrak{U}} \mathcal{F}(W\cap W').$$

We denote by $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ *the full subcategory of* $\mathrm{Psh}(X, \mathbf{CBorn}_K)$ *formed by sheaves.*

Example 24. *The structure sheaf* $\mathcal{O}_X$ *is a complete convex bornological sheaf on* X*. Indeed, recall that when we view* $\mathcal{O}_X$ *as a presheaf of Banach spaces, the algebraic exactness of the sequence*

$$0 \to \mathcal{F}(U) \to \prod_{V \in \mathfrak{U}} \mathcal{F}(V) \to \prod_{W, W' \in \mathfrak{U}} \mathcal{F}(W \cap W')$$

is Tate's Acyclicity Theorem, see for example [23, Theorem 8.2.1.1]*, and the topological exactness follows with the open mapping theorem. When we view* $\mathcal{O}_X$ *as a sheaf with values in* $\mathbf{CBorn}_K$*, the strictness follows directly from* Buchwalter's theorem [25, Theorem 4.9]*, which can be thought of as an analogue of the open mapping theorem in the bornological setting.*

The category of complete convex bornological sheaves is again well-behaved:

Lemma 25. *The category* $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ *is quasi-abelian.*

Proof. See [22, Proposition 2.2.7]. □

Note that the previous definitions also makes sense for presheaves $X_w^{\mathrm{op}} \to \mathbf{Ban}_K$, since every admissible covering is finite and $\mathbf{Ban}_K$ admits finite limits. But $\mathbf{Ban}_K$ is not cocomplete, and therefore the stalks of a sheaf of K-Banach spaces cannot be defined within $\mathbf{Ban}_K$. However, $\mathbf{CBorn}_K$ is cocomplete, which is why we can make the following

Definition 26. *Let* $\mathcal{F}$ *be a complete convex bornological presheaf on* X*, and let* $p \in \mathcal{P}(X)$ *denote a prime filter of* X*. The* stalk *of* $\mathcal{F}$ *at* p *is*

$$\mathcal{F}_p := \varinjlim_{U \in p} \mathcal{F}(U).$$

Example 27. *Consider the rigid analytic unit disc* $X = \mathbb{D}^1 = \mathrm{Sp}\, K\langle x \rangle$ *over* K*. The stalk of its structure sheaf at* 0 *is*

$$\mathcal{O}_{\mathbb{D}^1,0} = \varinjlim_{m \geq 0} K\left\langle \frac{x}{\pi^m} \right\rangle, \tag{1.1}$$

where $\pi \in K$ *denotes a uniformizer. Since all the structural morphisms*

$$K\left\langle \frac{x}{\pi^{m_1}} \right\rangle \to K\left\langle \frac{x}{\pi^{m_2}} \right\rangle \quad \text{for } m_1 \leq m_2$$

are injective, it follows with Lemma 17 that the colimit 1.1 can be calculated in $\mathbf{Born}_K$*. Now it follows with 12 that the underlying* K*-vector space of* $\mathcal{O}_{\mathbb{D}^1,0}$ *is given by the colimit of the the underlying* K*-vector spaces of the bornological spaces* $K\langle \frac{x}{\pi^m} \rangle$*, and a subset* $B \subseteq \mathcal{O}_{\mathbb{D}^1,0}$ *is bounded if and only if it is a bounded subset of some* $K\langle \frac{x}{\pi^m} \rangle$*.*

As for sheaves on abelian groups, we have a sheafification functor for complete convex bornological sheaves. We note that the construction of this functor relies on the cocompletness of $\mathbf{CBorn}_K$, and it is therefore not possible to get a similar result for presheaves with values in $\mathbf{Ban}_K$.

Lemma and Definition 28 ([22, Proposition 2.2.6])**.** *The inclusion functor*

$$\mathrm{Sh}(X, \mathbf{CBorn}_K) \hookrightarrow \mathrm{Psh}(X, \mathbf{CBorn}_K)$$

admits a left adjoint

$$\mathcal{F} \mapsto \mathcal{F}^{\mathrm{Sh}},$$

which turns $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ *into a relfective subcategory of* $\mathrm{Psh}(X, \mathbf{CBorn}_K)$*. Furthermore, consider the morphisms*

$$\mathrm{sh}_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F}^{\mathrm{Sh}}$$

for every complete convex bornological presheaf $\mathcal{F}$*, given by the unit of the adjunction. Then for every* $\mathcal{F} \in \mathrm{Psh}(X, \mathbf{CBorn}_K)$ *and every prime filter* $p \in \mathcal{P}(X)$*, the morphism of complete convex bornological* K*-vector spaces*

$$\mathrm{sh}_{\mathcal{F},p} \colon \mathcal{F}_p \to \mathcal{F}^{\mathrm{Sh}}_p$$

is an isomorphism.

For the remainder of this section, we discuss how the symmetric monoidal structure on $\mathbf{CBorn}_K$ lifts to a symmetric monoidal structure on $\mathrm{Sh}(X, \mathbf{CBorn}_K)$. These constructions again rely on the cocompleteness of $\mathbf{CBorn}_K$ and are therefore not possible for sheaves with values in $\mathbf{Ban}_K$.

First, we define a functor

$$\underline{\mathcal{H}om}_K \colon \mathrm{Psh}(X, \mathbf{CBorn}_K) \times \mathrm{Psh}(X, \mathbf{CBorn}_K) \to \mathrm{Psh}(X, \mathbf{CBorn}_K)$$

as follows. For any two complete convex bornological presheaves $\mathcal{F}$ and $\mathcal{G}$, consider the morphism

$$r \colon \prod_{U \in X_w} \underline{\mathrm{Hom}}_K\left(\mathcal{F}(U), \mathcal{G}(U)\right) \to \prod_{\substack{V,W \in X_w \\ W \subseteq V}} \underline{\mathrm{Hom}}_K\left(\mathcal{F}(V), \mathcal{G}(W)\right)$$

which sends $f = (f_U)_{U \in X_w}$ to the object $r(f)$ whose compoents are given by

$$r(f)_{V,W} = \sigma_{W \subseteq V} \circ f_V - f_W \circ \tau_{W \subseteq V}.$$

Here $\sigma_{W \subseteq V} \colon \mathcal{F}(V) \to \mathcal{F}(W)$ and $\tau_{W \subseteq V} \colon \mathcal{G}(V) \to \mathcal{G}(W)$ are the restriction maps. We denote the kernel of this morphism by

$$\underline{\mathrm{Hom}}_K(\mathcal{F}, \mathcal{G}) := \ker(r).$$

Then we introduce a complete convex bornological presheaf

$$\underline{\mathcal{H}om}_K\left(\mathcal{F}, \mathcal{G}\right) : U \mapsto \underline{\mathrm{Hom}}_K(\mathcal{F}|_U, \mathcal{G}|_U)$$

which turns out to be a sheaf when $\mathcal{F}$ and $\mathcal{G}$ are both sheaves. In particular, $\underline{\mathcal{H}om}_K$ restricts to a functor

$$\underline{\mathcal{H}om}_K \colon \mathrm{Sh}(X, \mathbf{CBorn}_K) \times \mathrm{Sh}(X, \mathbf{CBorn}_K) \to \mathrm{Sh}(X, \mathbf{CBorn}_K).$$

Next, we define for any two complete convex bornological presheaves $\mathcal{F}$ and $\mathcal{G}$ a presheaf

$$\mathcal{F} \widehat{\otimes}_{K,\mathrm{Psh}} \mathcal{G}$$

via

$$U \mapsto \mathcal{F}(U) \widehat{\otimes}_K \mathcal{G}(U).$$

Clearly, this defines a functor

$$-\widehat{\otimes}_{K,\mathrm{Psh}}- \colon \mathrm{Psh}(X, \mathbf{CBorn}_K) \times \mathrm{Psh}(X, \mathbf{CBorn}_K) \to \mathrm{Psh}(X, \mathbf{CBorn}_K).$$

Finally, let $\underline{K}_{X,\mathrm{Psh}}$ denote the constant presheaf on X with value K. One then shows that $\mathrm{Psh}(X, \mathbf{CBorn}_K)$ admits a closed symmetric monoidal structure, endowed with $\widehat{\otimes}_{K,\mathrm{Psh}}$ as internal tensor product, the constant sheaf $\underline{K}_{X,\mathrm{Psh}}$ as unit and $\underline{\mathcal{H}om}_K$ as internal homomorphism functor. Now it is easy to get a closed symmetric monoidal structure on $\mathrm{Sh}(X, \mathbf{CBorn}_K)$, by using the sheafification functor. Define for any two complete convex bornological presheaves $\mathcal{F}$ and $\mathcal{G}$ the sheaf

$$\mathcal{F}\widehat{\otimes}_K\mathcal{G} := \left(\mathcal{F}\widehat{\otimes}_{K,\mathrm{Psh}}\mathcal{G}\right)^{\mathrm{Sh}}$$

as well as the constant sheaf

$$\underline{K}_X := \left(\underline{K}_{X,\mathrm{Psh}}\right)^{\mathrm{Sh}}.$$

Theorem 29 ([22, Corollary 2.2.17]). *The category* $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ *endowed with* $\widehat{\otimes}_K$ *as internal tensor product,* $\underline{K}_X$ *as unit and* $\underline{\mathcal{H}om}_K$ *as internal homomorphism functor is a closed category.*

1.6. **Bounded linear endomorphisms of rigid analytic functions.** Having this new machinery of bornological spaces at hand, we now again discuss the goal of the author's research project: Developing an anologue of the classical Riemann-Hilbert correspondence for $\widehat{\mathcal{D}}$-modules on rigid analytic spaces. The point of depature of the author's work is not the classical proof of the classical Riemann-Hilbert correspondence, as for example in [17, Section 7]. Instead, we follow a generalisation of the Riemann-Hilbert correspondence to arbitrary perfect complexes of $\mathcal{D}_X^\infty$ given by Prosmans-Schneiders [26].

We would like to explain their approach. First, Prosmans-Schneiders view any sheaf on a complex analytic manifold X as a sheaf of ind-Banach spaces. Since the category of ind-Banach spaces $\mathbf{Ind}(\mathbf{Ban}_{\mathbb{C}})$ is an elementary closed quasi-abelian category, all the constructions in Schneiders article [22] apply. Now one views $\mathcal{O}_X$ in the derived category $\mathbf{D}(\mathrm{Sh}, \mathbf{Ind}(\mathbf{Ban}_{\mathbb{C}}))$ as a bimodule-object over the commutative algebra object $\mathcal{R}_X := \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{Ind}(\mathbf{Ban}_{\mathbb{C}})}(\mathcal{O}_X, \mathcal{O}_X)$ and the constant sheaf $\underline{\mathbb{C}}_X$. One then has a natural functos between triangulated categories

$$\begin{aligned} \mathbf{D}^b(\mathcal{R}_X)^{\mathrm{op}} &\to \mathbf{D}^b(\underline{\mathbb{C}}_X), \\ \mathcal{M} &\mapsto \mathbb{R}\underline{\mathcal{H}om}_{\mathcal{R}_X}(\mathcal{M}, \mathcal{O}_X) \end{aligned} \tag{1.2}$$

and a functor in the opposite direction

$$\begin{aligned} \mathbf{D}^b(\underline{\mathbb{C}}_X) &\to \mathbf{D}^b(\mathcal{R}_X)^{\mathrm{op}}, \\ \mathcal{M} &\mapsto \mathbb{R}\underline{\mathcal{H}om}_{\underline{\mathbb{C}}_X}(\mathcal{M}, \mathcal{O}_X). \end{aligned}$$

Now the key-technical result is that the canonical morphism

$$\mathcal{D}^\infty \xrightarrow{\cong} \mathcal{R}_X$$

is an isomorphism. We note that the statement that $\mathcal{D}^\infty$ coincides as a sheaf of Fréchet spaces with the continuous endomorphisms of the structure sheaf is due to Ishimura [27], and Prosmans-Schneiders achievement is the vanishing of the ext sheaves

$$\underline{\mathcal{E}xt}^i_{\mathbf{Ind}(\mathbf{Ban}_{\mathbb{C}})}(\mathcal{O}_X, \mathcal{O}_X) = 0 \text{ for } i > 0.$$

Using this result, one shows that the first functor 1.2 above defines a fully faithful embedding from the derived category of sheaves of perfect $\mathcal{D}_X^\infty$-modules into some derived category of sheaves of ind-Banach spaces on X.

The author aims to replicate Prosmans and Schneiders work in the rigid-analytic setting. Ardakov-Ben-Bassat [28] have already made a first step towards this goal: They established a version of Ishimuras result mentioned above for $\widehat{\mathcal{D}}$-modules. In what follows, we give an overview of their work.

Assume that K is a non-trivially valued, non-Archimedean valuation field of characteristic zero. Let X be a smooth rigid K-analytic space. Instead of working with sheaves of ind-Banach spaces or Fréchet spaces over K, Ardakov and Ben-Bassat work with sheaves valued in the category of complete convex bornological K-vector spaces $\mathbf{CBorn}_K$, which we have discussed in details in the previous subsection 1.5. This category admits an internal homomorphism functor $\underline{\mathcal{H}om}_K$, and we introduce the notation

$$\mathcal{E}_X := \underline{\mathcal{H}om}_K\left(\mathcal{O}_X, \mathcal{O}_X\right).$$

Ardakov and Ben-Bassat then introduce a morphism

$$\rho\colon \widehat{\mathcal{D}}_X \to \mathcal{E}_X.$$

of complete convex bornological sheaves. Here is the main result of their article.

Theorem 30 ([28, Theorem 3.8 and Lemma 4.20]). *Let K be a field of characteristic zero, complete with respect to a non-trivial non-Archimedean valuation. Then the following are equivalent:*

(i) *The homomorphism of sheaves of bornological K-vector spaces*

$$\rho\colon \widehat{\mathcal{D}}_X \to \mathcal{E}_X.$$

is an isomorphism of sheaves of bornological K-vector spaces for all smooth rigid K-analytic spaces X.

(ii) *The ground field K is algebraically closed, and its residue field k is uncountable.*

We would like to give an overview of their proof of direction (ii) $\Rightarrow$ (i). In order to do so, assume for the rest of this section that K has characteristic zero. We first give a concrete description of the sheaf $\mathcal{E}_X$. Recall that for every admissible open $U \in X_w$

$$\mathcal{E}_X(X) = \underline{\mathcal{H}om}_K\left(\mathcal{O}_X, \mathcal{O}_X\right)(U) = \underline{\mathrm{Hom}}_K\left(\mathcal{O}_U, \mathcal{O}_U\right).$$

Lemma 31. *For any two complete convex bornological sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, the underlying K-vector space of $\underline{\mathrm{Hom}}_K(\mathcal{F}, \mathcal{G})$ is given by the set of collections of bounded linear maps from $\mathcal{F}(U)$ to $\mathcal{G}(U)$ indexed by $U \in X_w$ that are compatible with the restriction. A subset $B \subseteq \underline{\mathrm{Hom}}_K(\mathcal{F}, \mathcal{G})$ is bounded if every projection to each complete convex bornological K-vector space $\underline{\mathrm{Hom}}_K\left(\mathcal{F}(U), \mathcal{G}(U)\right)$ is bounded.*

Proof. See the discussion in [28, Subsection 2.2]. □

It is not possible to give a concrete global description of $\widehat{\mathcal{D}}_X$. However, since Theorem 30 is a statement about a morphism of sheaves, it is enough to work locally.

Proposition 32 ([28, Proposition 4.15]). *Any smooth rigid analytic space X admits an admissible covering of affinoids X_i such that there exists an étale morphism $g_i\colon X_i \to \mathbb{D}^{n_i}$ to disks of various dimensions.*

Thus we can assume until the end of this subsection that X is a smooth affinoid K-variety equipped with an étale morphism

$$g\colon X \to \mathbb{D}^d = \operatorname{Sp} K\langle x_1, \dots, x_d\rangle.$$

Recall the following notion from functional analysis. We say that a family of vectors $(a_\alpha)_{\alpha\in\mathbb{N}^d}$ in a K-Banach space V is *rapidly decreasing* if

$$\pi^{-r|\alpha|}a_\alpha \to 0 \text{ as } |\alpha| \to \infty \text{ for any } r \in \mathbb{N}.$$

Lemma 33. *Denote the canonical lifts of vector fields* $\frac{d}{dx_i} \in \mathcal{T}(\mathbb{D}^d)$ *along* g *by* $\partial_i \in \mathcal{T}(X)$. *Then*

$$\widehat{\mathcal{D}}_X(X) = \left\{ \sum_{\alpha\in\mathbb{N}^d} a_\alpha \partial^\alpha \colon a_\alpha \in \mathcal{O}(X) \text{ and } (a_\alpha)_{\alpha\in\mathbb{N}^d} \text{ is rapidly decreasing} \right\}.$$

Moreover, the Fréchet structure on $\widehat{\mathcal{D}}_X(X)$ *can be defined by the family of seminorms*

$$\left| \sum_{\alpha\in\mathbb{N}^d} a_\alpha \partial^\alpha \right|_R = \sup_{\alpha\in\mathbb{N}^d} |a_\alpha| r^\alpha$$

for sufficiently large real numbers r. *Finally, this Fréchet structure induces the bornology on* $\widehat{\mathcal{D}}_X(X)$.

Proof. This is [28, Lemma 3.4]. For the bornology, see [28, Lemma 4.20]. □

Note that the previous Lemma 33 gives a complete characterisation of the sheaf $\widehat{\mathcal{D}}_X$. Indeed, for any admissible open $U \in X_w$, the composition $U \hookrightarrow X \xrightarrow{g} \mathbb{D}^d$ is still étale. Therefore, a similar characterisation of $\widehat{\mathcal{D}}_X(U)$ applies.

These concrete local characterisations of both $\widehat{\mathcal{D}}_X$ and $\mathcal{E}_X$ allow us to define locally an inverse of ρ_X, which we will now introduce. For any $\alpha \in \mathbb{N}^d$ write $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, $\alpha! = \alpha_1! \cdots \alpha_d!$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$. Write $\alpha \leq \beta$ to abbreviate $\alpha_i \leq \beta_i$ for $i = 1, \dots, d$ and set $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Moreover, for any $U \in X_w$ write x^α to denote the image of $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ in $\mathcal{O}_X(U)$ under the composition $\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d) \xrightarrow{g^\#} \mathcal{O}_X(X) \to \mathcal{O}_X(U)$. With this baggage of notation we now define for any $U \in X_w$, $\psi \in \operatorname{End}_K \mathcal{O}_X(U)$ and $\alpha \in \mathbb{N}^d$

$$\eta_\alpha(\psi) := \frac{1}{\alpha!} \sum_{\beta\leq\alpha} \psi(x^\beta) \binom{\alpha}{\beta} (-x)^{\alpha-\beta} \in \mathcal{O}_X(U).$$

Ardakov-Ben-Bassat then prove the following key result: If the residue field k of K is algebraically closed and uncountable, then the family

$$(\eta_\alpha(\phi(X)))_{\alpha\in\mathbb{N}^d} \subseteq \mathcal{O}_X(X)$$

is rapidly decreasing for every $\phi \in \mathcal{E}(X)$. Thus we can define, with the appropriate restrictions on the ground field, a morphism of $\mathcal{O}_X$-modules

$$\eta_X \colon \mathcal{E}_X \to \widehat{\mathcal{D}}_X$$

via

$$\eta_X(U)(\phi) := \sum_{\alpha\in\mathbb{N}^d} \eta_\alpha(\phi(U))\partial^\alpha \text{ for any } \phi \in \mathcal{E}_X(U).$$

This morphism turns out to be a morphism of sheaves of complete convex bornological K-vector spaces.

Ardakov-Ben-Bassat then prove via a concrete computation that

$$\eta_X \circ \rho_X = \mathrm{id}_{\widehat{\mathcal{D}}_X}.$$

When K is algebraically closed, one shows

$$\rho_X \circ \eta_X = \mathrm{id}_{\mathcal{E}_X},$$

not with a concrete calculation but slightly indirect. Recall, that we were working so far with a smooth affinoid K-variety X equipped with an étale morphism $g\colon X \to \mathbb{D}^d$. As mentioned above, we can apply 32 to get the global statement.

1.7. **Derived bounded linear endomorphisms of rigid analytic functions.** Since $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ is a quasi-abelian category, it admits a well-behaved derived category $\mathbf{D}(\mathrm{Sh}(X, \mathbf{CBorn}_K))$. Inspired by Prosmans-Schneiders and Ardakov-Ben-Bassat's work, the author aims to answer the following question in his research.[1]

Question 34. *Is the morphism*

$$\widehat{\mathcal{D}}_X \to \mathbb{R}\underline{\mathcal{H}om}_K(\mathcal{O}_X, \mathcal{O}_X)$$

induced by ρ_X an isomorphism in the derived category of sheaves of bornological K-vector spaces $\mathbf{D}(Sh(X, \mathbf{CBorn}_K))$? Equivalently, do we have

$$\underline{\mathcal{E}xt}^i_K(\mathcal{O}_X, \mathcal{O}_X) = 0 \text{ for all } i > 0?$$

In the following Section 2, we start tackling Question 34 by investigating a slightly different version of ρ_X. We call this morphism $\widetilde{\rho}_X$, and prove that it is an isomorphism if the ground field K is algebraically closed. This is analogous to Theorem 30, but the necessary restrictions on K are much milder. We see that $\widetilde{\rho}_X$ provides a new viewpoint on ρ_X in Section 3. In particular, Theorem 66 shifts our focus away from ρ_X along $\widetilde{\rho}_X$ towards the complete convex bornological $\mathcal{O}_X$-dual of a certain $\mathcal{O}_X$-linear morphism φ_X. In the remainder ot this section, we then work towards a derived version of Theorem 66. This encouters making Question 34 precise, as it is not a priori clear how to derive the internal homomorphism functor $\underline{\mathcal{H}om}_K$. Here a new ingredient comes in, namely the *left heart* of the category of (sheaves of) complete convex bornological K-vector spaces. In the final Section 4, we close this text with an outlook on the future research that the author aims to do.

2. Formal neighborhoods of the diagonal

For the remainder ot this article, we fix a field K of charcteristic zero, complete with respect to non-trivial non-Archimedean valution. X denotes a smooth rigid K-analytic space.

In this section, we aim to establish an analogue of Ardakov-Ben-Bassat's main result 30. First, we introduce sheaves of complete convex bornological sheaves of $\mathcal{O}_X$-modules. We describe the closed symmetric monoidal structure on the category of these objects, as well as a sheafification functor for complete convex bornological presheaves of $\mathcal{O}_X$-modules. Let $\Delta\colon X \to X \times X$ denote the diagonal morphism.

[1] We remark that its formulation is slightly sloppy since we have not explained yet how to derive the internal homomorphism functor $\underline{\mathcal{H}om}_K$. For the precise formulation, see Question 89.

Having the internal homomorphism functor over $\mathcal{O}_X$ in our disposal, we then define a morphism

$$\widetilde{\rho}_X \colon \widehat{\mathcal{D}}_X \to \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X\right).$$

Here, $\mathcal{O}_X$ acts on the first factor of $\Delta^{-1}\mathcal{O}_{X\times X}$. Then, we prove the following result, analogous to Theorem 30.

Theorem 35. *Let K be algebraically closed. Then the morphism*

$$\widetilde{\rho}_X \colon \widehat{\mathcal{D}}_X \to \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X\right)$$

is an isomorphism of complete convex bornological $\mathcal{O}_X$-modules.

We remark that the previous Theorem 35 does not need any additional restrictions on the ground field K, in contrast to Theorem 30.

2.1. **Bornological modules and algebras.** Recall that for any closed symmetric monoidal category $\mathbf{E}$ we have a category of commutative algebras $\mathbf{Comm}(\mathbf{E})$. Furthermore, every commutative algebra $R \in \mathbf{Comm}(\mathbf{E})$ gives rise to a category of R-modules $\mathbf{Mod}(R)$. It turns out that these categories of module objects again admit the structure of a closed symmetric monoidal category.

Lemma 36 ([24, Lemma 2.3])**.** *Suppose that $\mathbf{E}$ is a closed symmetric monoidal additive category with all finite limits and colimits and $R \in \mathbf{Comm}(\mathbf{E})$. Then $\mathbf{Mod}(R)$ is a closed symmetric monoidal category with all finite limits and colimits as well. These limits and colimits can be computed in $\mathbf{E}$.*

For future reference, we summarize the construction of the closed symmetric monoidal structure in $\mathbf{Mod}(R)$. Pick two R-modules M, N, and let $\mathrm{a}_M \colon R\otimes M \to M$ and $\mathrm{a}_N \colon R \otimes N \to N$ denote the action morphisms. Then $\underline{\mathrm{Hom}}_R(M,N)$ is defined as the limit of the diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(M,N) & \xrightarrow{\ f\mapsto f\circ \mathrm{a}_M\ } & \underline{\mathrm{Hom}}(R\otimes M, N) \\ & {\scriptstyle h\mapsto \mathrm{id}_R\otimes h}\searrow \quad \nearrow{\scriptstyle g\mapsto \mathrm{a}_N\circ g} & \\ & \underline{\mathrm{Hom}}(R\otimes M, R\otimes N). & \end{array} \tag{2.1}$$

Notice that we can define

$$L_{M,N} \colon R\otimes \underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(M,N)$$

as the composition

$$R\otimes \underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(M,M)\otimes \underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(M,N),$$

where $R \to \underline{\mathrm{Hom}}(M,M)$ is adjoint the action morphism $R\otimes M \to M$. One then shows that $L_{M,N}$ induces a well defined morphism on $\underline{\mathrm{Hom}}_R(M,N)$, which gives it the structure of an R-module. To define the monoidal structure, let $R\otimes N \to N\otimes R$ denote the symmetric structure. Then $M\otimes_R N$ is defined to be the element of $\mathbf{E}$ given as the colimit of the diagram

$$M\otimes R\otimes N \underset{(\mathrm{a}_M\otimes \mathrm{id}_F)\circ(\sigma\otimes \mathrm{id}_N)}{\overset{\mathrm{id}_M\otimes \mathrm{a}_N}{\rightrightarrows}} M\otimes N \tag{2.2}$$

equipped with the obvious action of R.

In this subsection, we aim to analyse the case $\mathbf{E} = \mathrm{Sh}(X; \mathbf{CBorn}_K)$. Let us start with modules and algebras in $\mathbf{CBorn}_K$.

Proposition 37. *The category* $\mathbf{Comm}(\mathbf{CBorn}_K)$ *of* complete convex bornological K-algebras *has the following characterisation. Its objects are commutative unitial K-algebras R equipped with a bornology $\mathcal{B}_R$, such that the underlying bornological K-vector space is complete convex, and such that the multiplication is bounded in the sense that for all bounded subsets $B_1, B_2 \in \mathcal{B}_R$, $B_1 B_2$ is bounded, too. Its morphisms are homomorphisms of rings that are bounded with respect to the bornologies.*

Proof. Straightforward from the definition of $\mathbf{Comm}(\mathbf{CBorn}_K)$. □

Proposition 38. *Fix a complete convex bornological K-algebra R. Consider the category* $\mathbf{Mod}(R)$ *of* complete convex bornological R-modules.

(i) *The objects in $\mathbf{Mod}(R)$ are R-modules M equipped with a bornology $\mathcal{B}_M$, such that the underlying bornological K-vector space is complete convex, and such that the scalar multiplication is bounded in the sense that for all bounded subsets $B_R \in \mathcal{B}_R$ and $B_M \in \mathcal{B}_M$, $B_R B_M$ is bounded, too. Its morphisms are homomorphisms of R-modules that are bounded with respect to the bornologies.*

(ii) $\mathbf{Mod}(R)$ *is quasi-abelian.*

(iii) *The forgetful functor*

$$\mathbf{Mod}(R) \to \mathbf{CBorn}_K$$

preserves limits and colimits. In particular, a morphism of $\mathbf{Mod}(R)$ is strict if and only if it is strict as a moprhism of $\mathbf{CBorn}_K$.

(iv) $\mathbf{Mod}(R)$ *admits a closed symmetric monoidal structre induced by the closed structure on $\mathbf{Born}_K$.*

Proof. (i) can be checked from the definition in a straightforward manner. The statements in (ii) and (iii) can be found in [22, Proposition 1.5.1], and (iv) is [24, Lemma 2.3]. □

Proposition 39. *Fix a complete convex bornological K-algebra R. For any two complete convex bornological R-modules M and N, $\underline{\mathrm{Hom}}_R(M,N)$ is given as follows. Its underlying K-vector space is the set of all R-linear maps $M \to N$. The bornology is the equiboundedness bornology. The action of R is given for all $r \in R$ and $f \in \underline{\mathrm{Hom}}_R(M,N)$ by*

$$(rf)(m) := rf(m) \text{ for all } m \in M.$$

Proof. The statement about the underlying vector space of $\underline{\mathrm{Hom}}_R(M,N)$ follows, because the underlying vector space of a limit of complete convex bornological vector spaces is given as the limit of the underlying vector spaces, see Lemma 16 (i). Next, note that $\underline{\mathrm{Hom}}_R(M,N)$ is given as the kernel of a bounded linear map $\underline{\mathrm{Hom}}_K(M,N) \to \underline{\mathrm{Hom}}_K(R \widehat{\otimes}_K M, N)$. It follows with [22, Remark 1.1.2 (a)] that

$$\underline{\mathrm{Hom}}_R(M,N) \to \underline{\mathrm{Hom}}_K(M,N)$$

is a natural morphism in $\mathbf{CBorn}_K$. Thus the bornology on $\underline{\mathrm{Hom}}_R(M,N)$ is induced by the bornology on $\underline{\mathrm{Hom}}_K(M,N)$, which is the equiboundedness bornology. It remains to prove the statement about the action of R on $\underline{\mathrm{Hom}}_R(M,N)$. However, this is straightforward to check. □

Next, we consider modules and algebras in $\mathrm{Sh}(X, \mathbf{CBorn}_K)$.

Proposition 40. *The category* $\mathbf{Comm}(\mathrm{Sh}(X, \mathbf{CBorn}_K))$ *of* sheaves of complete convex bornological K-algebras *has the following characterisation. Its objects are functors*

$$X_w^{\mathrm{op}} \to \mathbf{Comm}(\mathbf{CBorn}_K)$$

such that the composition with the forgetful functor $\mathbf{Comm}(\mathbf{CBorn}_K) \to \mathbf{CBorn}_K$ *is a sheaf of complete convex bornological K-vector spaces. Its morphisms are natural transformations.*

Proof. Straightforward from the definition of $\mathbf{Comm}(\mathrm{Sh}(X, \mathbf{CBorn}_K))$. □

Proposition 41. *Fix a sheaf of complete convex bornological K-algebras $\mathcal{R}$. Consider the category* $\mathbf{Mod}(\mathcal{R})$ *of* sheaves of complete convex bornological $\mathcal{R}$-modules.

(i) *Its objects are sheaves $\mathcal{F}$ with values in $\mathbf{CBorn}_K$, such that for each admissible open $U \in X_w$, $\mathcal{F}(U)$ is a bornological $\mathcal{R}(U)$-module, and for each inclusion of admissible opens $V \subseteq U$, the restriction morphism* $\mathrm{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ *is compatible with the module structures via the restriction morphism $\mathcal{R}(U) \to \mathcal{R}(V)$. To be precise, the latter condition means that for any inclusion of admissible opens $V \subseteq U$, the diagram*

$$\begin{array}{ccc} \mathcal{R}(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \big\downarrow & & \big\downarrow \\ \mathcal{R}(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commutes. A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of complete convex bornological $\mathcal{R}$-modules is a morphism of sheaves, such that for each admissible open $U \in X_w$, the morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{R}(U)$-modules.

(ii) $\mathbf{Mod}(\mathcal{R})$ *is quasi-abelian.*

(iii) *The forgetful functor*

$$\mathbf{Mod}(\mathcal{R}) \to \mathrm{Sh}(X, \mathbf{CBorn}_K)$$

preserves limits and colimits. In particular, a morphism of $\mathbf{Mod}(\mathcal{R})$ *is strict if and only if it is strict as a moprhism of* $\mathrm{Sh}(X, \mathbf{CBorn}_K)$.

(iv) $\mathbf{Mod}(\mathcal{R})$ *admits a closed symmetric monoidal structre induced by the closed structure on* $\mathrm{Sh}(X, \mathbf{CBorn}_K)$.

Proof. (i) can be checked from the definition in a straigtforward manner. The statements in (ii) and (iii) can be found in [22, Proposition 1.5.1], and (iv) is [24, Lemma 2.3]. □

We are now going to describe the internal homomophism functor $\underline{\mathcal{H}om}_{\mathcal{R}}$ in the category of sheaves of complete convex bornological $\mathcal{R}$-modules. We use the following notation. For any admissible open $U \in X_w$ write

$$\underline{\mathrm{Hom}}_{\mathcal{R}|_U}(\mathcal{F}|_U, \mathcal{G}|_U) := \underline{\mathcal{H}om}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})(U).$$

Proposition 42. *Fix a sheaf of complete convex bornological K-algebras $\mathcal{R}$. For any two sheaves of complete convex bornological $\mathcal{R}$-modules $\mathcal{F}$ and $\mathcal{G}$, the complete convex bornological $\mathcal{R}(X)$-module $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ is given as follows. Its underlying*

$\mathcal{R}(X)$-module is the set of all $\mathcal{R}$-linear maps $\mathcal{F} \to \mathcal{G}$. A subset is bounded if its projection to each complete convex bornological $\mathcal{R}(V)$-module $\underline{\mathrm{Hom}}_{\mathcal{R}(V)}(\mathcal{F}(V), \mathcal{G}(V))$ is bounded. The action of $\mathcal{R}(X)$ is given for all $r \in \mathcal{R}(X)$ and $f \in \underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ by

$$(rf)(U) := r(f(U)) \quad \textit{for all } U \in X_w.$$

Proof. Since a limit of sheaves can be calculated valueswise, $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ is the limit of the diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_K(\mathcal{F}, \mathcal{G}) & \xrightarrow{f \mapsto f \circ \mathrm{a}_{\mathcal{F}}} & \underline{\mathrm{Hom}}_K(\mathcal{R}\widehat{\otimes}_K \mathcal{F}, \mathcal{G}) \\ & {\scriptstyle h \mapsto \mathrm{id}_{\mathcal{R}} \widehat{\otimes}_K h} \searrow \qquad \nearrow {\scriptstyle g \mapsto \mathrm{a}_{\mathcal{G}} \circ g} & \\ & \underline{\mathrm{Hom}}_K(\mathcal{R}\widehat{\otimes}_K \mathcal{F}, \mathcal{R}\widehat{\otimes}_K \mathcal{G}), & \end{array}$$

where $\mathrm{a}_{\mathcal{F}}$ and $\mathrm{a}_{\mathcal{G}}$ denote the action morphisms $\mathcal{R}\widehat{\otimes}_K \mathcal{F} \to \mathcal{F}$ and $\mathcal{R}\widehat{\otimes}_K \mathcal{G} \to \mathcal{G}$. Now everything follows with similar arguments as in the proof of proposition 39, and the chracterisation of $\underline{\mathrm{Hom}}_K$ in Lemma 31. □

2.2. **Sheafification of sheaves of modules.** In the following subsections we often work with sheaves of modules that are defined as the sheafification of presheaves of modules. Therefore, we have to develop a sheafification functor for presheaves of complete convex bornological $\mathcal{R}$-modules. Since our results are true more generally, we first work with the category of presheaves $\mathrm{Psh}(X, \mathbf{E})$ and the category of sheaves $\mathrm{Sh}(X, \mathbf{E})$ valued in some elementary quasi-abelian symmetric monoidal category $\mathbf{E}$. We note that all facts stated in subsection 1.5 remain true in this setting: The categories of presheaves and sheaves with values in $\mathbf{E}$ are still well-defined, such presheaves admit stalks in $\mathbf{E}$, we have a sheafification functor $-^{\mathrm{Sh}}$ which gives rise to morphisms $\mathrm{sh}_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F}$ for every presheaf $\mathcal{F} \in \mathrm{Psh}(X, \mathbf{E})$ that are isomorphisms at the stalks, and the symmetric monoidal structure on $\mathbf{E}$ lifts to symmetric monoidal structures on $\mathrm{Psh}(X, \mathbf{E})$ and $\mathrm{Sh}(X, \mathbf{E})$.

Lemma 43. *Consider for any two objects $\mathcal{F}, \mathcal{G} \in \mathrm{Psh}(X, \mathbf{E})$ the natural morphism*

$$\gamma_{\mathcal{F},\mathcal{G}} \colon \mathcal{F} \otimes_{\mathrm{Psh}} \mathcal{G} \xrightarrow{\mathrm{sh}_{\mathcal{F}} \otimes_{\mathrm{Psh}} \mathrm{sh}_{\mathcal{G}}} \mathcal{F}^{\mathrm{sh}} \otimes_{\mathrm{Psh}} \mathcal{G}^{\mathrm{sh}} \xrightarrow{\mathrm{sh}_{\mathcal{F}^{\mathrm{sh}} \otimes_{\mathrm{Psh}} \mathcal{G}^{\mathrm{sh}}}} \mathcal{F}^{\mathrm{sh}} \otimes \mathcal{G}^{\mathrm{sh}}.$$

Its stalks are isomorphisms.

Proof. Pick an arbitrary prime filter $p \in \mathcal{P}$. Clearly, $\left(\mathrm{sh}_{\mathcal{F}^{\mathrm{sh}} \otimes_{\mathrm{Psh}} \mathcal{G}^{\mathrm{sh}}}\right)_p$ is an isomorphism, so it remains to investigate the stalk of $\mathrm{sh}_{\mathcal{F}} \otimes_{\mathrm{Psh}} \mathrm{sh}_{\mathcal{G}}$ at p.

First, we compute

$$\begin{aligned} (\mathrm{sh}_{\mathcal{F}} \otimes \mathrm{sh}_{\mathcal{G}})_p &= \varinjlim_{U \in X_w} \mathrm{sh}_{\mathcal{F}}(U) \otimes \mathrm{sh}_{\mathcal{G}}(U) \\ &= \varinjlim_{U, V \in X_w} \mathrm{sh}_{\mathcal{F}}(U) \otimes \mathrm{sh}_{\mathcal{G}}(V), \end{aligned}$$

since the set of pairs (U, U) with $U \in X_w$ is cofinal in the set of of pairs (U, V) with $U, V \in X_w$. Next, we have

$$\varinjlim_{U, V \in X_w} \mathrm{sh}_{\mathcal{F}}(U) \otimes \mathrm{sh}_{\mathcal{G}}(V) = \varinjlim_{U \in X_w} \varinjlim_{V \in X_w} \mathrm{sh}_{\mathcal{F}}(U) \otimes \mathrm{sh}_{\mathcal{G}}(V)$$

by [29, Tag 002M]. Since $\mathbf{C}$ is a closed category, every functor of the form $C \otimes -$ where $C \in \mathbf{C}$ is a left adjoint. Thus these functors commute with colimits. This shows

$$(\mathrm{sh}_{\mathcal{F}} \otimes_{\mathrm{Psh}} \mathrm{sh}_{\mathcal{G}})_p = \mathrm{sh}_{\mathcal{F},p} \otimes_{\mathrm{Psh}} \mathrm{sh}_{\mathcal{G},p},$$

which clearly is an isomorphism, because $\mathrm{sh}_{\mathcal{F},p}$ and $\mathrm{sh}_{\mathcal{G},p}$ are. □

Note that the previous Lemma 43 and its proof imply that

$$\gamma^{\mathrm{Sh}}_{\mathcal{F},\mathcal{G}} \colon (\mathcal{F} \otimes_{\mathrm{Psh}} \mathcal{G})^{\mathrm{Sh}} \to \mathcal{F}^{\mathrm{sh}} \otimes \mathcal{G}^{\mathrm{sh}}$$

is a functorial isomorphism. We denote its inverse by

$$\delta_{\mathcal{F},\mathcal{G}} \colon \mathcal{F}^{\mathrm{sh}} \otimes \mathcal{G}^{\mathrm{sh}} \to (\mathcal{F} \otimes_{\mathrm{Psh}} \mathcal{G})^{\mathrm{Sh}}.$$

We now have the following sheafification functor.

Proposition 44. *Let $\mathcal{R}$ be a commutative unitial algebra in $\mathrm{Psh}(X, \mathbf{E})$. Then the functor*

$$\begin{aligned} -^{\mathrm{Sh}} \colon \mathbf{Mod}(\mathcal{R}) &\to \mathbf{Mod}(\mathcal{R}^{\mathrm{Sh}}) \\ (\mathcal{F}, \mathrm{a}_{\mathcal{F}}) &\mapsto (\mathcal{F}^{\mathrm{Sh}}, \mathrm{a}^{\mathrm{Sh}}_{\mathcal{F}} \circ \delta_{\mathcal{R},\mathcal{F}}) \end{aligned}$$

is left adjoint to the inclusion functor

$$\begin{aligned} -^{\mathrm{Psh}} \colon \mathbf{Mod}(\mathcal{R}^{\mathrm{Sh}}) &\to \mathbf{Mod}(\mathcal{R}) \\ (\mathcal{G}, \mathrm{a}_{\mathcal{G}}) &\mapsto (\mathcal{G}, \mathrm{a}_{\mathcal{G}} \circ \gamma_{\mathcal{R},\mathcal{G}}). \end{aligned}$$

Proof. The sheafification functor

$$-^{\mathrm{sh}} \colon \mathrm{Psh}(X, \mathbf{E}) \to \mathrm{Sh}(X, \mathbf{E}) \tag{2.3}$$

is monoidal, by Lemma 43. Thus both functors $-^{\mathrm{Sh}}$ and $-^{\mathrm{Psh}}$ are well-defined. The adjunction follows easily from the adjunction of the functor 2.3 with the inclusion $\mathrm{Sh}(X, \mathbf{E}) \hookrightarrow \mathrm{Psh}(X, \mathbf{E})$. □

The following Lemma explains how the closed structures on $\mathbf{Mod}(\mathcal{R})$ compares to the closed structure on $\mathbf{Mod}(\mathcal{R}^{\mathrm{Sh}})$ with respect to the sheafification functor.

Proposition 45. *Let $\mathcal{R}$ be a commutative unitial algebra in $\mathrm{Psh}(X, \mathbf{E})$, and $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}(\mathcal{R})$.*

(i) *The unit of the sheafification adjunction gives a functorial isomorphism*

$$(\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})^{\mathrm{Sh}} \simeq \mathcal{F}^{\mathrm{Sh}} \otimes_{\mathcal{R}^{\mathrm{Sh}}} \mathcal{G}^{\mathrm{Sh}}$$

of $\mathcal{R}$-module objects.

(ii) *Suppose that $\mathcal{R}$ and $\mathcal{G}$ are sheaves. Then we have a functorial isomorphism*

$$\underline{\mathcal{H}om}_{\mathcal{R}}(\mathcal{F}, \mathcal{G}) \simeq \underline{\mathcal{H}om}_{\mathcal{R}}(\mathcal{F}^{\mathrm{Sh}}, \mathcal{G}).$$

of $\mathcal{R}$-module objects.

Proof. (i) Recall that the monoidal structure in a category of module objects is given by the colimit of some diagram 2.2. We compute

$$\begin{aligned}(\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})^{\mathrm{Sh}} &= \left(\varinjlim \left(\mathcal{F} \otimes_{\mathrm{Psh}} \mathcal{R} \otimes_{\mathrm{Psh}} \mathcal{G} \rightrightarrows \mathcal{F} \otimes_{\mathrm{Psh}} \mathcal{G} \right) \right)^{\mathrm{Sh}} \\ &= \varinjlim \left((\mathcal{F} \otimes_{\mathrm{Psh}} \mathcal{R} \otimes_{\mathrm{Psh}} \mathcal{G})^{\mathrm{Sh}} \rightrightarrows (\mathcal{F} \otimes_{\mathrm{Psh}} \mathcal{G})^{\mathrm{Sh}} \right) \\ &\overset{43}{=} \varinjlim \left(\mathcal{F}^{\mathrm{Sh}} \otimes \mathcal{R}^{\mathrm{Sh}} \otimes \mathcal{G}^{\mathrm{Sh}} \rightrightarrows \mathcal{F}^{\mathrm{Sh}} \otimes \mathcal{G}^{\mathrm{Sh}} \right) \\ &= \mathcal{F}^{\mathrm{Sh}} \otimes_{\mathcal{R}^{\mathrm{Sh}}} \mathcal{G}^{\mathrm{Sh}}.\end{aligned}$$

Clearly, all the steps in the previous computation commute with the $\mathcal{R}$-actions.

(ii) Compute

$$\begin{aligned}\mathrm{Hom}_{\mathcal{R}}\left(-, \underline{\mathcal{H}om}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})\right) &= \mathrm{Hom}_{\mathcal{R}}\left(- \otimes_{\mathcal{R}} \mathcal{F}, \mathcal{G}\right) \\ &\overset{44}{=} \mathrm{Hom}_{\mathcal{R}^{\mathrm{Sh}}}\left((- \otimes_{\mathcal{R}} \mathcal{F})^{\mathrm{Sh}}, \mathcal{G}\right) \\ &\overset{\text{(i)}}{=} \mathrm{Hom}_{\mathcal{R}^{\mathrm{Sh}}}\left(-^{\mathrm{Sh}} \otimes_{\mathcal{R}^{\mathrm{Sh}}} \mathcal{F}^{\mathrm{Sh}}, \mathcal{G}\right) \\ &= \mathrm{Hom}_{\mathcal{R}^{\mathrm{Sh}}}\left(-^{\mathrm{Sh}}, \underline{\mathcal{H}om}_{\mathcal{R}^{\mathrm{Sh}}}\left(\mathcal{F}^{\mathrm{Sh}}, \mathcal{G}\right)\right) \\ &= \mathrm{Hom}_{\mathcal{R}}\left(-, \underline{\mathcal{H}om}_{\mathcal{R}^{\mathrm{Sh}}}\left(\mathcal{F}^{\mathrm{Sh}}, \mathcal{G}\right)\right)\end{aligned}$$

and apply the Yoneda Lemma. □

We now specialise to the case $\mathbf{E} = \mathbf{CBorn}_K$. The characterisations of algebras and modules in $\mathrm{Psh}(X, \mathbf{CBorn}_K)$ are easier than the characterisations for sheaves. First, one checks

$$\mathbf{Comm}\left(\mathrm{Psh}\left(X, \mathbf{CBorn}_K\right)\right) = \mathbf{Fun}\left(X_w, \mathbf{Comm}\left(\mathbf{CBorn}_K\right)\right).$$

Let $\mathcal{R} \in \mathbf{Comm}\left(\mathrm{Psh}\left(X, \mathbf{CBorn}_K\right)\right)$. The verficiation of the following description of $\mathbf{Mod}(\mathcal{R})$ is straigtforward. Its objects are presheaves $\mathcal{F}$ with values in $\mathbf{CBorn}_K$, such that for each admissible open $U \in X_w$, $\mathcal{F}(U)$ is a bornological $\mathcal{R}(U)$-module, and for each inclusion of admissible opens $V \subseteq U$, the restriction morphism $\mathrm{res}_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the restriction morphism $\mathcal{R}(U) \to \mathcal{R}(V)$. To be precise, the latter condition means that for any inclusion of admissible opens $V \subseteq U$, the diagram

$$\begin{array}{ccc}\mathcal{R}(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{R}(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V)\end{array}$$

commutes. A morphism $\mathcal{F} \to \mathcal{G}$ of presheaves of complete convex bornological $\mathcal{R}$-modules is a morphism of presheaves, such that for each admissible open $U \in X_w$, the morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{R}(U)$-modules.

The description of the internal homomorphism functor in the category of presheaves of complete convex bornological $\mathcal{R}$ is basically the same as for the category of sheaves of complete convex bornological $\mathcal{R}^{\mathrm{Sh}}$-modules. Again, we use the notation

$$\underline{\mathrm{Hom}}_{\mathcal{R}|_U}\left(\mathcal{F}|_U, \mathcal{G}|_U\right) := \underline{\mathcal{H}om}_{\mathcal{R}}\left(\mathcal{F}, \mathcal{G}\right)(U)$$

for any admissible open $U \in X_w$.

Proposition 46. *Fix a presheaf of complete convex bornological K-algebras $\mathcal{R}$. For any two presheaves of complete convex bornological $\mathcal{R}$-modules $\mathcal{F}$ and $\mathcal{G}$, the complete convex bornological $\mathcal{R}(X)$-module $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ is given as follows. Its underlying $\mathcal{R}(X)$-module is the set of all $\mathcal{R}$-linear maps $\mathcal{F} \to \mathcal{G}$. A subset is bounded if its projection to each complete convex bornological $\mathcal{R}(V)$-module $\underline{\mathrm{Hom}}_{\mathcal{R}(V)}(\mathcal{F}(V), \mathcal{G}(V))$ is bounded. The action of $\mathcal{R}(X)$ is given for all $r \in \mathcal{R}(X)$ and $f \in \underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ by*

$$(rf)(U) := r(f(U)) \quad \text{for all } U \in X_w.$$

Proof. Similar to the proof of Proposition 42 □

For future reference, we state the following instance of Proposition 45.

Proposition 47. *Fix a sheaf of complete convex bornological K-algebras $\mathcal{R}$. Let $\mathcal{F}$ be a presheaf of complete convex bornological $\mathcal{R}$-modules, and let $\mathcal{G}$ denote a sheaf of complete convex bornological $\mathcal{R}$-modules. Then we have a functorial isomorphism*

$$\underline{\mathcal{H}om}_{\mathcal{R}}(\mathcal{F}^{\mathrm{Sh}}, \mathcal{G}) \simeq \underline{\mathcal{H}om}_{\mathcal{R}}(\mathcal{F}, \mathcal{G}),$$

of sheaves of complete convex bornological $\mathcal{R}$-modules.

2.3. **The presheaf $\Delta^{-1}_{\mathrm{Psh}}\mathcal{O}_{X\times X}$.** Recall that we aim to prove Theorem 35. In order to do so, we need a more concrete description of the sheaf of complete convex bornological $\mathcal{O}_X$-modules

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X).$$

Recall that $\Delta^{-1}\mathcal{O}_{X\times X}$ is defined to be the sheafification of the presheaf of complete convex bornological $\mathcal{O}_X$-modules

$$\Delta^{-1}_{\mathrm{Psh}}\mathcal{O}_{X\times X} \colon U \mapsto \varinjlim_{\substack{V \in (X\times X)_w \\ V \supseteq \Delta(U)}} \mathcal{O}_{X\times X}(V),$$

where $\mathcal{O}_X$ acts on the first factor. Proposition 47 implies that it is actually enough to work with this presheaf. This is good news, because it turns out that one can describe it very explicitly, when we work locally.

Therefore, we now assume until the end this subsection that X is a smooth affinoid K-variety equipped with an étale morphism $g \colon X \to \mathbb{D}^d$. First, the constraint that X is affinoid allows us the apply the following result, which we cite from [30, Lemma 2.3].

Lemma 48. *Let $X = \mathrm{Sp}\, A$ be an affinoid rigid space and let Z be a Zariski closed subspace. Then Z corresponds to an ideal $I = (f_1, \ldots, f_n)$. If U is any admissable open of X containing Z, then there exists an $n \geq 0$ such that*

$$Z \subseteq X\left(\frac{f_1, \ldots, f_n}{\pi^n}\right) \subseteq U.$$

In particular, the presheaf $\Delta^{-1}\mathcal{O}_{X\times X}$ is equal to the presheaf

$$U \mapsto \varinjlim_{m\geq 0} \mathcal{O}_{X\times X}(U\times U)\left\langle \frac{f_1|_{U\times U},\dots,f_r|_{U\times U}}{\pi^m}\right\rangle,$$

where $f_1,\dots,f_r$ generate the kernel of the multiplication morphism

$$\mathrm{mult}_{\mathcal{O}_X(X)}\colon \mathcal{O}_X(X)\widehat{\otimes}_K\mathcal{O}_X(X)\to\mathcal{O}_X(X).$$

This characterisation can be made more precise, using coordinates coming from the étale morphism $g\colon X\to\mathbb{D}^d$. To do so, we first cite the following list of notations with minor adaptions from [28, page 12]. We can lift the standard vector fields $\frac{d}{dx_1},\dots,\frac{d}{dx_n}\in\mathcal{T}(\mathbb{D}^d_k)$ along the étale morphism g to obtain $\partial_1,\dots,\partial_d\in\mathcal{T}(X)$. Note that we have $\mathcal{T}(X)=\bigoplus_{i=1}^d\mathcal{O}(X)\partial_i$. For every $\alpha\in\mathbb{N}^d$ we write $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$, $\alpha!=\alpha_1!\cdots\alpha_d!$, and $\binom{\alpha}{\beta}=\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_d}{\beta_d}$. We write $\alpha\leq\beta$ to mean that $\alpha_i\leq\beta_i$ for $i=1,\dots,d$, and we define $|\alpha|=\alpha_1+\cdots+\alpha_d$.

$\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d_K)=K\langle x_1,\dots,x_d\rangle$ denote the global sections of the d-dimensional unit disc. For any $U\in X_w$, we will abuse notation and write x^α to denote the image of $x^\alpha=x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ in $\mathcal{O}_X(U)$ under the composition $\mathcal{O}(\mathbb{D}^d)\xrightarrow{g^\#}\mathcal{O}_X(X)\to\mathcal{O}_X(U)$. Furthermore, we denote the global sections of the $2d$-dimensional unit by $\mathcal{O}_{\mathbb{D}^{2d}}(\mathbb{D}^{2d})=K\langle x_1,\dots,x_d,y_1,\dots,y_d\rangle$. For any $U\in(X\times X)_w$ we write y^α to denote the image of $y^\alpha=y_1^{\alpha_1}\cdots y_d^{\alpha_d}$ in $\mathcal{O}_{X\times X}(U)$ under the composition $\mathcal{O}(\mathbb{D}^{2d})\xrightarrow{(g\times g)^\#}\mathcal{O}_{X\times X}(X\times X)\to\mathcal{O}_{X\times X}(U)$.

We now characterise $\Delta^{-1}_{\mathrm{Psh}}\mathcal{O}_{X\times X}$ as follows. Denote for every integer $m\geq 0$ the complete convex bornological presheaf

$$U\mapsto\left(\mathcal{O}_X\widehat{\otimes}_{K,\mathrm{Psh}}\mathcal{O}_X\right)(U)\left\langle\frac{y-x}{\pi^m}\right\rangle$$

by $\left(\mathcal{O}_X\widehat{\otimes}_{K,\mathrm{Psh}}\mathcal{O}_X\right)\left\langle\frac{y-x}{\pi^m}\right\rangle$.

Proposition 49. *Suppose that K is algebraically closed. If X is connected, then* $\ker\mathrm{mult}_{\mathcal{O}_X(U)}$ *is generated by $\{y_1-x_1,\dots,y_d-x_d\}$ for every $U\in X_w$. Moreover, the natural morphism*

$$\varinjlim_{m\geq 0}\left(\mathcal{O}_X\widehat{\otimes}_{K,\mathrm{Psh}}\mathcal{O}_X\right)\left\langle\frac{y-x}{\pi^m}\right\rangle\to\Delta^{-1}_{\mathrm{Psh}}\mathcal{O}_{X\times X}$$

is an isomorphism.

We now work towards a proof of Proposition 49.

Lemma 50. *The morphism $g^\#(\mathbb{D}^d)$ is injective.*

Proof. The proof of [31][Proposition 8.8.1] shows that $g^\#(\mathbb{D}^d)$ is flat. Since $\mathcal{O}(\mathbb{D}^d)$ is a domain, we can apply [28][Lemma 4.1] to finish the proof. □

Lemma 51. *Suppose that K is algebraically closed. If X is connected, then*

$$g^\#\colon\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)\to\mathcal{O}_X(X)$$

has dense image.

Proof. By [28, Lemma 4.12], there exists an affinoid subdomain $t\colon Y \hookrightarrow X$ such that $(g \circ t)^{\#}\colon \mathcal{O}_{\mathbb{D}^d_K}(\mathbb{D}^d_K) \to \mathcal{O}_X(Y)$ has dense image. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{D}^d_K}(\mathbb{D}^d_K) & \xrightarrow{g^{\#}} & \mathcal{O}_X(X) \\ & \overset{(g\circ t)^{\#}}{\searrow} & \Big\downarrow t^{\#} \\ & & \mathcal{O}_X(Y), \end{array}$$

where the arrow at the right-hand side is injective because of [28, Proposition 4.2] and continous. In particular, $g^{\#}$ has dense image. $\square$

Lemma 52. *Let $u\colon E \to F$ denote a morphism between two K-Banach spaces with dense image. Then the morphism*

$$u \widehat{\otimes}_K u\colon E \widehat{\otimes}_K E \to F \widehat{\otimes}_K F$$

is has dense image, too.

Proof. A morphism in $\mathbf{Ban}_K$ is an epimorphism if and only if it has dense image, see for example [24, Lemma A.29(8)]. Since the symmetric monoidal structure on $\mathbf{Ban}_K$ is closed, the completed tensor product is a left adjoint and thus preserves epimorphisms. It follows that both morphisms

$$u \widehat{\otimes}_K \mathrm{id}_E\colon E \widehat{\otimes}_K E \to F \widehat{\otimes}_K E$$

and

$$\mathrm{id}_F \widehat{\otimes}_K u\colon F \widehat{\otimes}_K E \to F \widehat{\otimes}_K F$$

are epimorphisms, and so is their composition

$$u \widehat{\otimes}_K u\colon E \widehat{\otimes}_K E \to F \widehat{\otimes}_K F.$$

That is, it has dense image. $\square$

For the next lemma, we remark that the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{D}^{2d}}(\mathbb{D}^{2d}) & \xrightarrow{\mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d_K)}} & \mathcal{O}_{\mathbb{D}^d_K}(\mathbb{D}^d) \\ \Big\downarrow g^{\#}(\mathbb{D}^d)\widehat{\otimes} g^{\#}(\mathbb{D}^d) & & \Big\downarrow g^{\#}(\mathbb{D}^d) \\ \mathcal{O}_X(X) \widehat{\otimes}_K \mathcal{O}_X(X) & \xrightarrow{\mathrm{mult}_{\mathcal{O}_X(X)}} & \mathcal{O}_X(X) \end{array}$$

implies

$$(g^{\#}(\mathbb{D}^d) \widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d_K}(\mathbb{D}^d)}) \subseteq \ker \mathrm{mult}_{\mathcal{O}_X(X)} .$$

Lemma 53. *Suppose that K is algebraically closed and X is connected. Then* $(g^{\#}(\mathbb{D}^d) \widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)})$ *is dense in* $\ker \mathrm{mult}_{\mathcal{O}_X(X)}$.

Proof. Pick an arbitrary open subset $U \subseteq \mathcal{O}_X(X)$ such that

$$(g^{\#}(\mathbb{D}^d) \widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)}) \subseteq U \subsetneq \ker \mathrm{mult}_{\mathcal{O}_X(X)} .$$

Now fix an arbitrary $f \in U$. We have to show that f lies in $(g^{\#}(\mathbb{D}^d) \widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)})$.

Because of Lemma 51 and 52, it follows that $g^{\#} \widehat{\otimes}_K g^{\#}$ has dense image, and thus

$$U \subseteq g^{\#}(\mathbb{D}^d) \widehat{\otimes}_K g^{\#}(\mathbb{D}^d) \left(\mathcal{O}_{\mathbb{D}^{2d}}(\mathbb{D}^{2d})\right).$$

In particular, there exists an element $\tilde{f} \in \mathcal{O}_{\mathbb{D}^{2d}}(\mathbb{D}^{2d})$ such that $f = (g^{\#}(\mathbb{D}^d)\widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\tilde{f})$. Now the commutativity of the diagram 2.3 implies that

$$\mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)}(\tilde{f}) \in \ker g^{\#}(\mathbb{D}^d).$$

By Lemma 50, $g^{\#}(\mathbb{D}^d_K)$ is injective. Thus the previous statement implies

$$\mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)}(\tilde{f}) = 0$$

and therefore $\tilde{f} \in \ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)}$, which implies $f \in (g^{\#}(\mathbb{D}^d)\widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)})$. Since the choice of f was arbitrary, this proves

$$(g^{\#}(\mathbb{D}^d)\widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)}) = U.$$

Hence we win. □

Proof of Proposition 49. Since the restriction of $g\colon X \to \mathbb{D}^d$ to $U \subseteq X$ is still étale, it will be sufficient to consider in the proof of the first part of the proposition the case $U = X$ only. We introduce the notation

$$I := \langle y_1 - x_1, \ldots, y_d - x_d \rangle \subseteq \mathcal{O}_X(X)\widehat{\otimes}_K \mathcal{O}_X(X).$$

We have

$$(g^{\#}(\mathbb{D}^d)\widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)}) \subseteq I \subseteq \ker \mathrm{mult}_{\mathcal{O}_X(X)}.$$

By [31, Theorem 3.2.1,(3)], I is closed. Since $\ker \mathrm{mult}_{\mathcal{O}_X(X)}$ is closed, too, and $(g^{\#}(\mathbb{D}^d)\widehat{\otimes}_K g^{\#}(\mathbb{D}^d))(\ker \mathrm{mult}_{\mathcal{O}_{\mathbb{D}^d}(\mathbb{D}^d)})$ is dense in $\ker \mathrm{mult}_{\mathcal{O}_X(X)}$ by Lemma 53, it follows that $I = \ker \mathrm{mult}_{\mathcal{O}_X(X)}$.

We now want to prove the second half of the proposition. Using the same argument as above, it will be sufficient to prove that

$$\varinjlim_{m\geq 0} \left(\mathcal{O}_X \widehat{\otimes}_{K,\mathrm{Psh}} \mathcal{O}_X\right)(X) \left\langle \frac{y-x}{\pi^m} \right\rangle \to \Delta^{-1}_{\mathrm{Psh}} \mathcal{O}_{X\times X}(X)$$

is an isomorphism. Since $\ker \mathrm{mult}_{\mathcal{O}_X(X)}$ is generated by $\{y_1 - x_1, \ldots, y_d - x_d\}$, and K is affinoid, this follows from Lemma 48. □

We summarize the main consequence of this subsection in the following

Proposition 54. *Assume that X is an affinoid K-variety, equipped with an étale morphism $g\colon X \to \mathbb{D}^d$. Then we have an isomorphism*

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X\right) \simeq \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\varinjlim_{m\geq 0} \left(\mathcal{O}_X \widehat{\otimes}_{K,\mathrm{Psh}} \mathcal{O}_X\right) \left\langle \frac{y-x}{\pi^m} \right\rangle, \mathcal{O}_X\right)$$

of sheaves of complete convex bornological $\mathcal{O}_X$-modules.

We will use this identification in the future, without mentioning it further.

2.4. **The morphisms ζ_X and ϑ_X.** Endow $\underline{\mathcal{H}om}_K(\mathcal{O}_X, \mathcal{O}_X)$ with an $\mathcal{O}_X$-action as follows. For every two admissible opens $V \subseteq U$, $f \in \mathcal{O}_X(U)$ and $\phi \in \underline{\mathrm{Hom}}_K(\mathcal{O}_U, \mathcal{O}_U)$, define $(f\phi)(V)$ via

$$s \mapsto f \cdot (\phi(V)(s)) \text{ for all } s \in \mathcal{O}_X(V).$$

This turns $\underline{\mathcal{H}om}_K(\mathcal{O}_X, \mathcal{O}_X)$ into a complete convex bornological $\mathcal{O}_X$-module, and it turns ρ_X into a morphism of complete convex bornological $\mathcal{O}_X$-modules.

Now we introduce the following two morphisms of complete convex bornological $\mathcal{O}_X$-modules. Since they are defined via completed tensor products and composition with morphisms of complete convex bornological sheaves, they are again morphisms of complete convex bornological sheaves. The $\mathcal{O}_X$-linearity follows with an easy computation.

Definition 55. *We define two morphisms of complete convex bornological $\mathcal{O}_X$-modules.*

(i) *First, we introduce*

$$\zeta_X \colon \underline{\mathcal{H}om}_K(\mathcal{O}_X, \mathcal{O}_X) \to \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X, \mathcal{O}_X)$$

via

$$\zeta_X(U)(\phi) := \operatorname{mult}_{\mathcal{O}_U} \circ \left(\operatorname{id}_{\mathcal{O}_U} \widehat{\otimes}_K \phi\right)$$

for all $U \in X_w$ and $\phi \in \underline{\operatorname{Hom}}_K(\mathcal{O}_U, \mathcal{O}_U)$.

(ii) *Next, we have*

$$\vartheta_X \colon \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X, \mathcal{O}_X) \to \underline{\mathcal{H}om}_K(\mathcal{O}_X, \mathcal{O}_X)$$

via

$$\vartheta_X(U)(\phi) := \phi \circ \left(1_{\mathcal{O}_X} \widehat{\otimes}_K \operatorname{id}_{\mathcal{O}_X}\right)$$

for all $U \in X_w$ and $\phi \in \underline{\mathcal{H}om}_{\mathcal{O}_U}(\mathcal{O}_U \widehat{\otimes}_K \mathcal{O}_U, \mathcal{O}_U)$. Here,

$$1_{\mathcal{O}_X} \colon \underline{K}_X \to \mathcal{O}_X$$

denotes the 1 of the commutative unitial algebra $\mathcal{O}_X$.

We have the following result.

Lemma 56. *The morpisms ζ_X and ϑ_X are two-sided inverses and thus isomorphisms of complete convex bornological $\mathcal{O}_X$-modules.*

Proof. This is again an easy computation. □

2.5. **The morphism $\tilde{\rho}_X$.** In this subsection, we aim to define an $\mathcal{O}_X$-linear morphism

$$\tilde{\rho}_X \colon \widehat{\mathcal{D}}_X \to \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X \times X}, \mathcal{O}_X\right).$$

First we introduce the following $\mathcal{O}_X$-linear morphism. Consider the presheaf

$$\mathcal{O}_X \widehat{\otimes}_{K,\mathrm{Psh}} \mathcal{O}_X \colon U \mapsto \mathcal{O}_X(U) \widehat{\otimes}_{K,\mathrm{Psh}} \mathcal{O}_X(U)$$

which we regard as a presheaf complete convex bornological $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is acting on the first factor. There is an obvious morphism of presheaves of complete convex bornological $\mathcal{O}_X$-modules

$$\varphi_{X,\mathrm{Psh}} \colon \mathcal{O}_X \widehat{\otimes}_{K,\mathrm{Psh}} \mathcal{O}_X \to \Delta^{-1}_{\mathrm{Psh}} \mathcal{O}_{X \times X},$$

and we denote its sheafification by

$$\varphi_X \colon \mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X \to \Delta^{-1} \mathcal{O}_{X \times X}.$$

Proposition 57. *Pick an admissible covering $\{X_i\}$ of X by affinoids equipped with étale morphisms $g_i \colon X_i \to \mathbb{D}^{d_i}$. Let $\phi \in \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X \widehat{\otimes} \mathcal{O}_X, \mathcal{O}_X)$ such that the family*

$$(\phi(X_i)((y-x)^\alpha))_{\alpha \in \mathbb{N}^d} \subseteq \mathcal{O}_X(X_i) \ \ \textit{for all } i$$

is rapidly decreasing.

(i) *Then there exists exactly one $\mathcal{O}_X$-linear morphism $\widetilde{\phi}\colon \Delta^{-1}\mathcal{O}_{X\times X} \to \mathcal{O}_X$, such that the diagram*

$$\begin{array}{ccc} \Delta^{-1}\mathcal{O}_{X\times X} & & \\ {\scriptstyle \varphi_X}\uparrow & \overset{\tilde{\phi}}{\searrow} & \\ \mathcal{O}_X\widehat{\otimes}\mathcal{O}_X & \xrightarrow{\phi} & \mathcal{O}_X \end{array} \tag{2.4}$$

commutes.

(ii) *Assume that X is itself affinoid and equipped with an étale moprhism $g\colon X \to \mathbb{D}^d$. If $B \subseteq \underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\mathcal{O}_X\widehat{\otimes}_K\mathcal{O}_X, \mathcal{O}_X\right)$ is a bounded subset of such morphisms ϕ, then the set $\widetilde{B} := \left\{\widetilde{\phi}\colon \phi \in B\right\}$ is bounded in $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X)$ too.*

Proof. Note that the Proposition 57 really is a statement about the precomposition with φ_X

$$\varphi_X^*\colon \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X\widehat{\otimes}\mathcal{O}_X, \mathcal{O}_X) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X).$$

Therefore, it is enough to prove the proposition for the presheaves of complete convex bornological $\mathcal{O}_X$-modules $\Delta_{\mathrm{Psh}}^{-1}\mathcal{O}_{X\times X}$ and $\mathcal{O}_X\widehat{\otimes}_{\mathrm{Psh},K}\mathcal{O}_X$, as well as the morphism $\varphi_{X,\mathrm{Psh}}$ of presheaves of complete convex bornological $\mathcal{O}_X$-modules, instead of their sheafifications.

(i) First, assume that X is itself affinoid and equppied with an étale morphism $g \in X \to \mathbb{D}^d$. Note that the morphisms

$$\left(\mathcal{O}_X\widehat{\otimes}_{\mathrm{Psh},K}\mathcal{O}_X\right)(U)\left\langle\frac{y-x}{\pi^{m_1}}\right\rangle \to \left(\mathcal{O}_X\widehat{\otimes}_{\mathrm{Psh},K}\mathcal{O}_X\right)(U)\left\langle\frac{y-x}{\pi^{m_2}}\right\rangle, \text{ for } m_1 \leq m_2,$$

are injective. Therefore, it follows with Lemma 17 that the colimit $\varinjlim_{m\geq 0}\left(\mathcal{O}_X\widehat{\otimes}_{\mathrm{Psh},K}\mathcal{O}_X\right)(U)\left\langle\frac{y-x}{\pi^m}\right\rangle$ can be calculated in $\mathbf{Born}_K$. Now we apply Lemma 12 (ii) to see that the underlying vector space of this colimit is equal to

$$\left\{\sum_{\alpha\in\mathbb{N}^d} a_\alpha\frac{(y-x)^\alpha}{\pi^{m|\alpha|}} : (a_\alpha)_{\alpha\in\mathbb{N}^d} \subseteq \mathcal{O}_X(U)\widehat{\otimes}_K\mathcal{O}_X(U) \text{ with } a_\alpha \to 0 \text{ for } |\alpha| \to \infty, \text{ and } m \geq 0\right\}.$$

Since $(\phi(X)((y-x)^\alpha))_{\alpha\in\mathbb{N}^d}$ is rapidly decreasing, it follows that $(\phi(U)((y-x)^\alpha))_{\alpha\in\mathbb{N}^d}$ is rapidly decreasing for all $U \in X_w$. Thus ϕ extends to $\varinjlim_{m\geq 0}\left(\mathcal{O}_X\widehat{\otimes}_{K,\mathrm{Psh}}\mathcal{O}_X\right)\left\langle\frac{y-x}{\pi^m}\right\rangle$ by continuity:

$$\widetilde{\phi}(U)\left(\sum_{\alpha\in\mathbb{N}^d} a_\alpha\frac{(y-x)^\alpha}{\pi^{m|\alpha|}}\right) := \sum_{\alpha\in\mathbb{N}^d}\phi(U)(a_\alpha)\frac{\phi(U)\left((y-x)^\alpha\right)}{\pi^{m|\alpha|}}$$

for all $U \in X_w$. Clearly, these morphisms commute with the restriction morphisms and thus give a well-defined morphism

$$\widetilde{\phi}\colon \Delta_{\mathrm{Psh}}^{-1}\mathcal{O}_{X\times X} \to \mathcal{O}_X$$

of presheaves of complete convex bornological $\mathcal{O}_X$-modules, which fits into the desired commutative diagram 2.4. Note that the uniqueness follows, since we defined $\tilde{\phi}$ via extension by continuity.

Next, consider the global setting in the statment of the Proposition 57. If C_i is a connected component of X_i, then the composition $C_i \hookrightarrow X_i \to \mathbb{D}^{d_i}$ is still étale and the family $(\phi(C_i)((y-x)^\alpha))_{\alpha\in\mathbb{N}^d}$ is rapidly decreasing in

$\mathcal{O}_X(C_i)$. Therefore, we can assume without loss of generality that every X_i is connected.

It now follows by the previous discussion that that every $\phi|_{X_i}$ extends along $\varphi_{X_i,\mathrm{Psh}}$ to an $\mathcal{O}_{X_i}$-linear morphism $\widetilde{\phi|_{X_i}}\colon \Delta_{\mathrm{Psh}}\mathcal{O}_{X_i\times X_i} \to \mathcal{O}_{X_i}$. Since these morphisms are defined via extension by continuity, and because of Proposition 49, it follows that they agree on intersections $X_i \cap X_j$. The uniqueness follows with the local uniqueness discussed above.

(ii) It remains to show the statement about the boundedness. Recall that $B \subseteq \underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\mathcal{O}_X\widehat{\otimes}_{K,\mathrm{Psh}}\mathcal{O}_X, \mathcal{O}_X\right)$ is bounded, if its projection to each $\mathcal{O}_X(U)$-module $\underline{\mathrm{Hom}}_{\mathcal{O}_X(U)}\left(\mathcal{O}_X(U)\widehat{\otimes}_{K,\mathrm{Psh}}\mathcal{O}_X(U), \mathcal{O}_X(U)\right)$ is bounded. Since the composition $U \hookrightarrow X \xrightarrow{g} \mathbb{D}^d$ is still étale, it remains to show the following

Lemma 58. *Suppose that X is affinoid and equipped with an étale morphism $g\colon X \to \mathbb{D}^d$. Let $D \subseteq \underline{\mathrm{Hom}}_{\mathcal{O}_X(X)}\left(\mathcal{O}_X(X)\widehat{\otimes}_K\mathcal{O}_X(X), \mathcal{O}_X(X)\right)$ be a bounded subset, such that for every $\psi \in D$, the family*

$$(\psi((y-x)^\alpha))_{\alpha\in\mathbb{N}^d} \subseteq \mathcal{O}_X(X)$$

is rapidly decreasing. Extend every element $\psi \in D$, as in the proof of Proposition 57, by continuity to an element

$$\widetilde{\phi} \in \underline{\mathrm{Hom}}_{\mathcal{O}_X(U)}\left(\varinjlim_{m\geq 0}\left(\mathcal{O}_X(X)\widehat{\otimes}_K\mathcal{O}_X(X)\right)\left\langle\frac{y-x}{\pi^m}\right\rangle, \mathcal{O}_X(X)\right).$$

Then $\widetilde{D} := \left\{\widetilde{\psi}\colon \psi \in D\right\}$ is bounded too.

Proof. We have to show that for every bounded subset

$$\widetilde{P} \subseteq \varinjlim_{m\geq 0}\left(\mathcal{O}_X(X)\widehat{\otimes}_K\mathcal{O}_X(X)\right)\left\langle\frac{y-x}{\pi^m}\right\rangle,$$

the subset

$$\widetilde{D}(\widetilde{P}) = \bigcup_{\tilde{\psi}\in\widetilde{D}} \tilde{\psi}(\widetilde{P}) \subseteq \mathcal{O}_X(X)$$

is bounded too. We know from the desricption of the colimit in $\mathbf{Born}_K$, that the boundedness of $\widetilde{P}$ implies that it is a bounded subset of some affinoid algebra $\left(\mathcal{O}_X(X)\widehat{\otimes}_K\mathcal{O}_X(X)\right)\left\langle\frac{y-x}{\pi^{m_0}}\right\rangle$. Denote the norm on this affinoid algebra by $\|-\|_{m_0}$, and suppose that $\widetilde{P}$ is bounded in this norm by a constant $C_1 > 0$. This means

$$\sup_{\alpha\in\mathbb{N}^d}\|a_\alpha\| = \left\|\sum_{\alpha\in\mathbb{N}^d} a_\alpha\frac{(y-x)^\alpha}{\pi^{m_0|\alpha|}}\right\|_{m_0} < C_1$$

for all $\sum_{\alpha\in\mathbb{N}^d} a_\alpha\frac{(y-x)^\alpha}{\pi^{m_0|\alpha|}} \in \widetilde{P}$. Therefore, the set $A \subseteq \mathcal{O}_X(X)\widehat{\otimes}_K\mathcal{O}_X(X)$ of all a_α that appear as a coefficient of an element of P is bounded. Since D is bounded, it follows that

$$D(A) = \bigcup_{\psi\in D}\psi(A) \subseteq \mathcal{O}_X(X)$$

is bounded, say by some constant $\widetilde{C}_1 > 0$. Next, note that since $(\psi((y - x)^\alpha))_{\alpha\in\mathbb{N}^d}$ is rapidly decreasing, there exists some constant $\widetilde{C}_2 > 0$ such that

$$\left\| \frac{\psi((y-x)^\alpha)}{\pi^{m_0|\alpha|}} \right\| < \widetilde{C}_2 \text{ for all } \alpha \in \mathbb{N}^d.$$

Now we compute for every $\widetilde{\psi} \in \widetilde{D}$ and $a = \sum_{\alpha\in\mathbb{N}^d} a_\alpha \frac{(y-x)^\alpha}{\pi^{m_0|\alpha|}} \in \widetilde{P}$

$$\left\|\widetilde{\psi}(a)\right\| = \left\| \sum_{\alpha\in\mathbb{N}^d} \psi(a_\alpha) \frac{\psi((y-x)^\alpha)}{\pi^{m_0|\alpha|}} \right\| = \sup_{\alpha\in\mathbb{N}^d} \|\psi(a_\alpha)\| \left\| \frac{\psi((y-x)^\alpha)}{\pi^{m_0|\alpha|}} \right\| < \widetilde{C}_1\widetilde{C}_2.$$

Thus $\widetilde{D}(\widetilde{P})$ is bounded. Hence we win. □

This proves Proposition 57 (ii).

□

The idea for the definition of ρ_X is as follows: We consider a differential operator $a \in \widehat{\mathcal{D}}_X(X)$ first as a morphism $\mathcal{O}_X \to \mathcal{O}_X$, then as an $\mathcal{O}_X$-linear morphism $\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X \to \mathcal{O}_X$, and finally we lift it along φ_X via Proposition 57. This is possible, because of the following

Lemma 59. *Suppose that X is affinoid and equipped with an étale morphism $g\colon X \to \mathbb{D}^d$. Pick a differential operator $a = \sum_{\alpha\in\mathbb{N}^d} a_\alpha \partial^\alpha \in \widehat{\mathcal{D}}_X(X)$, an admissible open $U \in X_w$, and set*

$$\phi := (\zeta_X \circ \rho_X)(U)(a) \in \underline{\mathrm{Hom}}_{\mathcal{O}_U}(\mathcal{O}_U \widehat{\otimes}_K \mathcal{O}_U, \mathcal{O}_U).$$

(i) *We compute*

$$\phi(X)\left((y-x)^\alpha\right) = \alpha! a_\alpha \textit{ for all } \alpha \in \mathbb{N}^d.$$

(ii) *The sequence*

$$\left(\phi(X)\left((y-x)^\alpha\right)\right)_{\alpha\in\mathbb{N}^d} \subseteq \mathcal{O}_X(U)$$

is rapidly decreasing.

Proof. Note that

$$(\zeta_X \circ \rho_X)(X)(a)\left((y-x)^\alpha\right) = \alpha! \eta_\alpha(\rho_X(X)(a)),$$

using the notation of [28]. Then (i) follows from the calculations made in [28, Proof of Lemma 4.11]. (ii) follows, since $|\alpha!| \le 1$ for all $\alpha \in \mathbb{N}^d$ and $(a_\alpha)_{\alpha\in\mathbb{N}^d}$ is rapidly decreasing. □

We now arrived at the main result of this subsection.

Lemma and Definition 60. *We introduce the morphism*

$$\widetilde{\rho}_X \colon \widehat{\mathcal{D}}_X \to \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X\right),$$

of complete convex bornological $\mathcal{O}_X$-modules which is given by

$$\widetilde{\rho}_X(V)(a) := \widetilde{(\zeta_X \circ \rho_X)(V)}(a) \tag{2.5}$$

for every $U \in X_w$ and $a \in \widehat{\mathcal{D}}_X(V)$, using the notation from Proposition 57.

Proof. A priori, $\widetilde{\rho}_X$ only is a morphism of sheaves of algebraic $\mathcal{O}_X$-modules. However, note that Proposition 57 locally sends bounded sets to bounded sets. Thus we can glue these morphisms together to get a morphism of sheaves of complete convex bornological $\mathcal{O}_X$-modules. □

2.6. **The morphism $\tilde{\eta}_X$.** In this subsection, we assume troughout that X is affinoid and equipped with an étale morphism $g\colon X \to \mathbb{D}^d$.

Because the characteristic of K is zero, we can make the following

Definition 61. *Define for any $U \in X_w$, $\psi \in \operatorname{Hom}_{\mathcal{O}_X(X)}\left(\Delta_{\mathrm{Psh}}^{-1}\mathcal{O}_{X\times X}(U), \mathcal{O}_X(U)\right)$ and $\alpha \in \mathbb{N}^d$*

$$\widetilde{\eta}_\alpha(\psi) := \frac{1}{\alpha!}\psi\left((y-x)^\alpha\right) \in \mathcal{O}_X(U).$$

The following proposition is the analogue of [28, Theorem 4.9] in our setting.

Proposition 62. *For every $\phi \in \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X)$, the family*

$$\left(\widetilde{\eta}_\alpha(\phi(X))\right)_{\alpha\in\mathbb{N}^d} \subseteq \mathcal{O}_X(X)$$

is rapidly decreasing.

Proof. By [28, Lemma 4.8], there exists a real number $0 < \varpi \leq 1$ depending only on K such that $\varpi^{|\alpha|} \leq |\alpha!|$ for all $\alpha \in \mathbb{N}^d$. We compute for every $r \in \mathbb{N}$

$$\begin{aligned}\left|\pi^{-r|\alpha|}\widetilde{\eta}_\alpha(\phi(X))\right| &= \left|\frac{1}{\alpha!}\psi\left(\pi^{-r|\alpha|}(y-x)^\alpha\right)\right| \\ &\leq \varpi^{|\alpha|}\left|\psi\left(\pi^{-r|\alpha|}(y-x)^\alpha\right)\right| \to 0 \text{ for } |\alpha| \to \infty,\end{aligned}$$

since the family $\left((y-x)^\alpha\right)_{\alpha\in\mathbb{N}^d}$ is rapidly decreasing in $\Delta_{\mathrm{Psh}}^{-1}\mathcal{O}_{X\times X}(X)$. □

Remark 63. *Recall that Ardakov-Ben-Bassat had to give a complicated proof as well as strong restrictions on the ground field K to establish their version* [28, Theorem 4.9] *of the previous Proposition 62. The reason for this is that they consider ϕ as a K-linar morphism $\mathcal{O}_X \to \mathcal{O}_X$, that is as an $\mathcal{O}_X$-linear morphism $\mathcal{O}_X\widehat{\otimes}_K\mathcal{O}_X \to \mathcal{O}_X$. In contrast, the author of this text embedds the sheaf $\mathcal{O}_X\widehat{\otimes}_K\mathcal{O}_X$ via φ_X into another sheaf $\Delta^{-1}\mathcal{O}_{X\times X}$. To illustrate the subtle difference between these sheaves, consider the unit disc $X = \mathbb{D}^1 = \operatorname{Sp} K\langle x\rangle$. Then the global sections of the presheaves of interest are*

$$\begin{aligned}(\mathcal{O}_{\mathbb{D}^1}\widehat{\otimes}_K\mathcal{O}_{\mathbb{D}^1})(\mathbb{D}^1) &= K\langle x, y\rangle, \text{ and} \\ (\Delta_{\mathrm{Psh}}^{-1}\mathcal{O}_{\mathbb{D}^2})(\mathbb{D}^1) &= \varinjlim_{m\geq 0} K\langle x, y, \frac{y-x}{\pi^m}\rangle.\end{aligned}$$

Note that

$$\varphi_{\mathbb{D}^1,\mathrm{Psh}}(\mathbb{D}^1)\colon K\langle x, y\rangle \to \varinjlim_{m\geq 0} K\langle x, y, \frac{y-x}{\pi^m}\rangle$$

is a bimorphism, that is it is a monomorphism and an epimorphism. Indeed, it is injective and thus its kernel is zero, and it is an epimorphism because its set-theoretic image is dense in the codomain. This illustrates how close both presheaves and both sheaves are to each other.

We summarise that intuitively, one should regard the sheaf $\Delta^{-1}\mathcal{O}_{X\times X}$ to be almost the same as $\mathcal{O}_X\widehat{\otimes}_K\mathcal{O}_X$, but ensuring that the sequence $\left((y-x)^\alpha\right)_{\alpha\in\mathbb{N}^d} \subseteq \Delta^{-1}\mathcal{O}_{X\times X}(X)$ is rapidly decreasing. We will see later in Theorem 66 that this new point of view shifts the focus away from ρ_X towards the $\mathcal{O}_X$-linear dual of φ_X.

The previous Proposition 62 enables us the make the following definition, without any additional restrictions on the ground field.

Definition 64. *We introduce a morphism of complete convex bornological $\mathcal{O}_X$-modules*

$$\widetilde{\eta}_X \colon \underline{\mathcal{H}om}_{\mathcal{O}_X}(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X) \to \widehat{\mathcal{D}}_X$$

via

$$\widetilde{\eta}_X(U)(\phi) := \sum_{\alpha\in\mathbb{N}^d} \widetilde{\eta}_\alpha(\phi(U))\partial^\alpha$$

for any $U \in X_w$ and $\phi \in \mathcal{E}_X(U)$.

Lemma 65. *For any $U \in X_w$, the morphism $\widetilde{\eta}_X(U)$ indeed sends bounded sets to bounded sets.*

Proof. The proof is identical to [28, First half of proof of Lemma 4.20]. Also, note that since $U \hookrightarrow X \to \mathbb{D}^d$ is still étale, it is enough to check that $\widetilde{\eta}_X(X)$ sends bounded sets to bounded sets.

We have to check for any B bounded in $\underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X\right)$ that $\tilde{\eta}_X(X)(B)$ is bounded in $\widehat{\mathcal{D}}_X(X)$. That is, we need to show that for each real number $R > 0$, there is a constant $C = C(R)$ such that

$$\sup_{\alpha\in\mathbb{N}^d} |\widetilde{\eta}_\alpha(X)(\phi)| r^\alpha < C$$

for all $\phi \in B$. Now $\{\phi(X)\colon \phi \in B\}$ is a bounded subset of $\underline{\mathrm{Hom}}_{\mathcal{O}_X(X)}\left(\Delta^{-1}_{\mathrm{Psh}}\mathcal{O}_{X\times X}(X), \mathcal{O}_X(X)\right)$ and so because the map

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X(X)}\left(\Delta^{-1}_{\mathrm{Psh}}\mathcal{O}_{X\times X}(X), \mathcal{O}_X(X)\right) \to \mathbb{R}_{\geq 0}$$

defined by

$$\phi \mapsto \sup_{\alpha\in\mathbb{N}^d} |\widetilde{\eta}_\alpha(X)(\phi)| r^\alpha = \sup_{\alpha\in\mathbb{N}^d} \left|\frac{1}{\alpha!}\phi(X)\left((y-x)^\alpha\right)\right| r^\alpha$$

is bounded, the image of $\{\phi(X)\colon \phi \in B\}$ is bounded and so we can find C as needed. □

2.7. **Proof of Theorem 35.** $\widehat{\mathcal{D}}_X$ and $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X)$ are sheaves on X_w. Therefore, the morphism $\widetilde{\rho}_X$ is an isomorphism if and only if its restriction to each member of an admissible affinoid covering of X is an isomorphism. By Proposition 32, we may therefore assume that X is an affinoid variety which admits an étale morphism $g\colon X \to \mathbb{D}^d$.

We aim to show that $\widetilde{\eta}_X$ and $\widetilde{\rho}_X$ are two-sided inverses. First, we check $\widetilde{\eta}_X \circ \widetilde{\rho}_X = \mathrm{id}_{\widehat{\mathcal{D}}_X}$ via direct computation: We have

$$\begin{aligned}\widetilde{\eta}_\alpha\left((\widetilde{\rho}_X)(X)(a)\right) &= \widetilde{\eta}_\alpha\left(\widetilde{(\zeta_X \circ \rho_X)(U)}(a)\right)\\ &= \eta_\alpha\left(\rho_X(U)(a)\right)\\ &= a_\alpha\end{aligned}$$

for any $U \in X_w$ and $a = \sum_{\alpha\in\mathbb{N}^d} a_\alpha\partial^\alpha \in \widehat{\mathcal{D}}_X(U)$, using the notation and calculation in the [28, Proof of Lemma 4.11]. This shows that $\widetilde{\eta}_X$ is a left inverse of $\widetilde{\rho}_X$.

Second, we have to check that $\widetilde{\rho}_X \circ \widetilde{\eta}_X$ is the identity on $\underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{O}_X\right)$. We see that

$$\widetilde{\eta}_X \circ (\widetilde{\rho}_X \circ \widetilde{\eta}_X - 1) = (\widetilde{\eta}_X \circ \widetilde{\rho}_X - 1) \circ \widetilde{\eta}_X = 0$$

Thus it will be enough to show that $\widetilde{\eta}_X$ is injective. Suppose that $\widetilde{\eta}_X(U)(\phi) = 0$ for some $U \in X_w$ and some $\mathcal{O}_X$-linear morphism $\Delta^{-1}\mathcal{O}_{X\times X} \to \mathcal{O}_X$. Let $\{V_i\}$ denote the admissible covering of U by its connected components; then also $\widetilde{\eta}_X(V_i)(\phi|_{V_i}) = \widetilde{\eta}_X(U)(\phi)|_{V_i} = 0$ and hence $\widetilde{\eta}_\alpha(\phi(V_i)) = 0$ for all i and $\alpha \in \mathbb{N}^d$. This implies

$$\eta_\alpha\left(\vartheta(V_i)\left(\phi(V_i)\circ\varphi_{V_i}\right)\right) = \widetilde{\eta}_\alpha(\phi(V_i)) = 0$$

for all i and $\alpha \in \mathbb{N}^d$. Since K is algebraically closed, one gets

$$\vartheta(V_i)\left(\phi(V_i)\circ\varphi_{V_i}\right) = 0$$

with [28, Proposition 4.13]. Recall that $\vartheta(V_i)$ is an isomorphism, see Lemma 56, and that the precomposition with φ_{V_i} is injective, see Proposition 57. It follows that $\phi(V_i) = 0$ for every i and thus $\phi(U) = 0$. This finishes the proof of Theorem 35. $\square$

3. ρ_X IS THE $\mathcal{O}_X$-DUAL OF φ

In the previous section, we introducted several morphisms $\widetilde{\rho}_X$, $\widetilde{\eta}_X$, ζ_X and φ_X. In this section, we fit all these objects in one coherent picture, and explain how the author aims to use this machinery to study the morphism ρ_X in the derived setting.

Here is the key observation.

Theorem 66.

(i) *All the previously defined morphisms fit into a commutative diagram*

$$\begin{array}{ccc} \underline{\mathcal{H}om}_{\mathcal{O}_X}(\Delta^{-1}\mathcal{O}_{X\times X},\mathcal{O}_X) & \xrightarrow{\varphi_X^*} & \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{O}_X\widehat{\otimes}_K\mathcal{O}_X,\mathcal{O}_X) \\ {\scriptstyle\widetilde{\rho}_X}\uparrow & & \uparrow{\scriptstyle\zeta_X} \\ \widehat{\mathcal{D}}_X & \xrightarrow{\qquad\rho_X\qquad} & \underline{\mathcal{H}om}_K(\mathcal{O}_X,\mathcal{O}_X) \end{array} \tag{3.1}$$

of complete convex bornological $\mathcal{O}_X$-modules. Furthermore, ζ_X is an isomorphism.

(ii) *Let K be algebraically closed. Then $\widetilde{\rho}_X$ is an isomorphism too. In particular, ρ_X is the complete convex bornological $\mathcal{O}_X$-dual of φ_X.*

Proof. We have to check that the diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathcal{O}_U}(\Delta^{-1}\mathcal{O}_{U\times U},\mathcal{O}_U) & \xrightarrow{\varphi_X^*(U)} & \underline{\mathrm{Hom}}_{\mathcal{O}_U}(\mathcal{O}_U\widehat{\otimes}_K\mathcal{O}_U,\mathcal{O}_U) \\ {\scriptstyle\tilde{\rho}_X(U)}\uparrow & & \uparrow{\scriptstyle\zeta_X(U)} \\ \widehat{\mathcal{D}}_X(U) & \xrightarrow{\qquad\rho_X(U)\qquad} & \underline{\mathrm{Hom}}_K(\mathcal{O}_U,\mathcal{O}_U) \end{array}$$

commutes for any admissible open $U \in X_w$. This follows directly from the definitions: We compute for every $a \in \widehat{\mathcal{D}}_X(U)$

$$\begin{aligned} (\varphi_X^*\circ\tilde{\rho}_X)(U)(a) &= \varphi_X^*(U)\left(\widetilde{(\zeta_X\circ\rho_X)(U)}(a)\right) \\ &= \widetilde{(\zeta_X\circ\rho_X)(U)}(a)\circ\varphi_U \\ &= (\zeta_X\circ\rho_X)(U)(a). \end{aligned}$$

The morphism ζ_X is an isomorphism by 56. When K is algebraically closed, $\widetilde{\rho}_X$ is an isomorphism by Theorem 35. $\square$

We aim to get a derived version of Theorem 66. However there is a technical difficulty: It is not know yet, whether the category $\mathrm{Sh}(X, \mathbf{CBorn}_K)$, or even $\mathbf{CBorn}_K$, has enough injectives. In fact, the author believes that there are not enough injective objects, and he is currently working towards a proof for this claim. However, there is a way around this issue, which we discuss in the following subsection.

3.1. **The derived category of a quasi-abelian category.** Recall that $\mathbf{CBorn}_K$ and $\mathrm{Sh}(\mathbf{CBorn}_K)$ are not abelian, but quasi-abelian categories. Schneiders [22] showed that any quasi-abelian category $\mathbf{E}$ admits a derived category, which is defined in a similar manner as the derived category of an abelian category.

Definition 67. *A complex $X^\bullet$ of $\mathbf{E}$ is* strictly exact *(resp.* coexact*) if for every i, the sequence*

$$X^{n-1} \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1}$$

is strictly exact *(resp.* coexact*), which means that d^{n-1} (resp. d^n) is strict and the canonical morphism*

$$\operatorname{im} d^{n-1} \to \ker d^n$$

is an isomorphism.

Let $\mathbf{K}(\mathbf{E})$ denote its category of complexes modulo homotopy, and let $\mathbf{N}(\mathbf{E})$ denote the full subcategory of the category $\mathbf{K}(\mathbf{E})$ of *strictly exact complexes.* One then shows that $\mathbf{N}(\mathbf{E})$ forms a saturated null system in $\mathbf{K}(\mathbf{E})$, which gives rise to the following

Definition 68 ([22, Definition 1.2.16])**.** *The* derived category *of $\mathbf{E}$ is*

$$\mathbf{D}(\mathbf{C}) := \mathbf{K}(\mathbf{E})/\mathbf{N}(\mathbf{E}).$$

A morphism of $\mathbf{K}(\mathbf{C})$ which has a strictly exact mapping cone is called a strict quasi-isomorphism*.*

In analogy to the abelian case, one can define what it means for an additive functor

$$\mathbf{F} \colon \mathbf{E}_1 \to \mathbf{E}_2$$

between two quasi-abelian categories to be *right* or *left derivable.* If $\mathbf{F}$ is right derivable, we denote its right derived functor by

$$\mathbb{R}\mathbf{F} \colon \mathbf{D}^+(\mathbf{E}_1) \to \mathbf{D}^+(\mathbf{E}_2).$$

If it is left derivable, we denote its left derived functor by

$$\mathbb{L}\mathbf{F} \colon \mathbf{D}^-(\mathbf{E}_1) \to \mathbf{D}^-(\mathbf{E}_2).$$

One computes derived functors via $\mathbf{F}$-injective or $\mathbf{F}$-projective subcategories of $\mathbf{E}_1$. They are defined as in the abelian case, except that we require certain morphisms in this definitions to be strict. We refer the reader to [22, Definition 1.3.2] for details. We say that $\mathbf{F}$ is *explicitly right* (resp. *left*) derivable if $\mathbf{E}_1$ has an $\mathbf{F}$-injective (resp. $\mathbf{F}$-projective) subcategory, and obtain the following result.

Proposition 69. *Let $\mathbf{E}_1, \mathbf{E}_2$ be quasi-abelian categories and let*

$$\mathbf{F} \colon \mathbf{E}_1 \to \mathbf{E}_2$$

be an additive functor.

(i) *Assume* $\mathbf{E}_1$ *has an* $\mathbf{F}$*-injective subcategory. Then* $\mathbf{F}$ *has a right derived functor*

$$\mathbb{R}\mathbf{F}\colon \mathbf{D}^+(\mathbf{E}_1) \to \mathbf{D}^+(\mathbf{E}_2).$$

(ii) *Assume* $\mathbf{E}_1$ *has an* $\mathbf{F}$*-projective subcategory. Then* $\mathbf{F}$ *has a left derived functor*

$$\mathbb{L}\mathbf{F}\colon \mathbf{D}^-(\mathbf{E}_1) \to \mathbf{D}^-(\mathbf{E}_2).$$

The proof of this proposition works in the abelian case.

We also note that functors can be derived via injective and projective objects. Recall that an object I of an abelian category is injective if $\mathrm{Hom}(-, I)$ is exact, and an object P of an abelian category is projective if $\mathrm{Hom}(P, -)$ is exact. Thus we have to introduce an analogue of exact functors in the quasi-abelian setting in order to establish a definition of injective and projective objects in quasi-abelian categories.

Definition 70. *An additive functor between to quasi-abelian categories*

$$\mathbf{F}\colon \mathbf{E}_1 \to \mathbf{E}_2$$

is

(i) left exact *if it transforms any strictly exact sequence*

$$0 \to E' \to E \to E'' \to 0$$

of $\mathbf{E}_1$ *into a strictly exact sequence*

$$0 \to \mathbf{F}(E') \to \mathbf{F}(E) \to \mathbf{F}(E'')$$

of $\mathbf{E}_2$*,*

(ii) strongly left exact *if it transforms any strictly exact sequence*

$$0 \to E' \to E \to E''$$

of $\mathbf{E}_1$ *into a strictly exact sequence*

$$0 \to \mathbf{F}(E') \to \mathbf{F}(E) \to \mathbf{F}(E'')$$

of $\mathbf{E}_2$*,*

(iii) right exact *if it transforms any strictly coexact sequence*

$$0 \to E' \to E \to E'' \to 0$$

of $\mathbf{E}_1$ *into a strictly coexact sequence*

$$\mathbf{F}(E') \to \mathbf{F}(E) \to \mathbf{F}(E'') \to 0$$

of $\mathbf{E}_2$*,*

(iv) strongly right exact *if it transforms any strictly coexact sequence*

$$E' \to E \to E'' \to 0$$

of $\mathbf{E}_1$ *into a strictly coexact sequence*

$$\mathbf{F}(E') \to \mathbf{F}(E) \to \mathbf{F}(E'') \to 0$$

of $\mathbf{E}_2$*,*

(v) exact *if it transforms any strictly exact sequence*

$$0 \to E' \to E \to E'' \to 0$$

of $\mathbf{E}_1$ *into a strictly exact sequence*

$$0 \to \mathbf{F}(E') \to \mathbf{F}(E) \to \mathbf{F}(E'') \to 0$$

of $\mathbf{E}_2$*, and*

(vi) strictly exact *(resp.* coexact*) if it transforms any strictly exact (resp. coexact) sequence*

$$0 \to E' \to E \to E'' \to 0$$

of $\mathbf{E}_1$ *into a strictly exact (resp. coexact) sequence*

$$\mathbf{F}(E') \to \mathbf{F}(E) \to \mathbf{F}(E'')$$

of $\mathbf{E}_2$*.*

Now we can make the following

Definition 71. *An object I of $\mathbf{E}$ is* injective *(resp.* strongly injective*) if the functor*

$$\mathrm{Hom}_{\mathbf{E}}(-, I)\colon \mathbf{E}^{\mathrm{op}} \to \mathbf{Ab}$$

is exact (resp. strongly exact). Dually, an object P of $\mathbf{E}$ is projective *(resp.* strongly projective*) if the functor*

$$\mathrm{Hom}_{\mathbf{E}}(P, -)\colon \mathbf{E} \to \mathbf{Ab}$$

is exact (resp. strongly exact).

We refer to [22, Section 1.3] for many more information on derived functors of additive functors between quasi-abelian categories. The reason why the author of this text provided the bare minimum is that the derived category of a quasi-abelian category $\mathbf{E}$ is actually equivalent to the derived category of an abelian category, namely its left heart $\mathbf{LH}(\mathbf{E})$, which we explain in the following subsection.

3.2. **The left heart of a quasi-abelian category.** Let $\mathbf{E}$ denote a quasi-abelian category. Its derived category $\mathbf{D}(\mathbf{E})$ admits a canonical t-structure, called the *left t-structure of* $\mathbf{D}(\mathbf{E})$. The *left heart* of $\mathbf{E}$

$$\mathbf{LH}(\mathbf{E})$$

is the heart of this t-structure. In this subsection, we explain some properties of this construction as well as its significance for the homological algebra of the quasi-abelian category $\mathbf{E}$.

First, we note that the t-structure on $\mathbf{D}(\mathbf{E})$ gives cohomology functors

$$LH^i\colon \mathbf{D}(\mathbf{E}) \to \mathbf{LH}(\mathbf{E}),$$

which can be characterised as follows.

Lemma 72 ([22, Corollary 1.2.20])**.** *The functor LH^i sends a complex $X^\bullet \in \mathbf{D}(\mathbf{E})$ to the complex*

$$0 \to \operatorname{coim} d^{i-1} \to \ker d^i \to 0$$

with $\ker d^i$ *in degree* 0*. In particular, the cohomology object $LH^i(X^\bullet)$ vanishes if and only if $X^\bullet$ is strictly exact in degree i.*

One can also give a concrete description of the left heart...

Lemma 73 ([22, Corollary 1.2.21])**.** *The left heart of* $\mathbf{E}$ *is equivalent to the localisation of the full subcategory of* $\mathbf{K}(\mathbf{E})$ *consisting of complexes* $X^\bullet$ *of the form*

$$0 \to X^{-1} \overset{\delta_{X^\bullet}}{\to} X^0 \to 0$$

(X^0 *in degree 0,* $\delta_{X^\bullet}$ *monomorphism) by the multiplicative system formed by morphisms* $f\colon X^\bullet \to Y^\bullet$ *such that the square*

$$\begin{array}{ccc} Y^{-1} & \xrightarrow{\delta_{Y^\bullet}} & Y^0 \\ {\scriptstyle u^{-1}}\uparrow & & {\scriptstyle u^0}\uparrow \\ X^{-1} & \xrightarrow{\delta_{X^\bullet}} & X^0 \end{array}$$

is both cartesian and cocartesian.

...and relate it the original category $\mathbf{E}$ as follows. Denote by

$$\mathbf{I}\colon \mathbf{E} \to \mathbf{LH}(\mathbf{E})$$

the canonical functor which sends an object X of $\mathbf{E}$ to the complex

$$0 \to X \to 0$$

with X in degree 0.

Lemma 74.

(i) *The functor*

$$\mathbf{C}\colon \mathbf{LH}(\mathbf{E}) \to \mathbf{E}$$

which send a complex $X^\bullet$ *to the cokernel of* $\delta_{X^\bullet}$ *is well-defined, and left adjoint to* $\mathbf{I}$*.*

(ii) $\mathbf{I}$ *makes* $\mathbf{E}$ *a reflexive subcategory of* $\mathbf{LH}(\mathbf{E})$*.*

(iii) $\mathbf{I}$ *is fully faithful.*

(iv) *A sequence*

$$X' \to X \to X''$$

is strictly exact in $\mathbf{E}$ *if and only if the sequence*

$$\mathbf{I}(X)' \to \mathbf{I}(X) \to \mathbf{I}(X)''$$

is exact in $\mathbf{LH}(\mathbf{E})$*.*

Proof. (i) and (ii) are [22, Proposition 1.2.27]. (iii) and (iv) are [22, Proposition 1.2.28]. □

The following result is the reason why we care about the left heart.

Proposition 75 ([22, Proposition 1.2.32])**.** *The canonical embedding*

$$\mathbf{I}\colon \mathbf{E} \to \mathbf{LH}(\mathbf{E})$$

induces an equivalence of categories

$$\mathbf{D}(\mathbf{I})\colon \mathbf{D}(\mathbf{E}) \overset{\cong}{\to} \mathbf{D}(\mathbf{LH}(\mathbf{E}))$$

which exchanges the left t-structure of $\mathbf{D}(\mathbf{E})$ *with the usual t-structure of* $\mathbf{D}(\mathbf{LH}(\mathbf{E}))$*.*

Recall that $\mathbf{CBorn}_K$ probably does not have enough injectives. In particular, it might not be possible to derived the internal homomorphism functor in $\mathrm{Sh}(X, \mathbf{CBorn}_K)$. Now the interpretation of the previous Proposition 75 is as follows: Instead of deriving the internal homomorphism functor in $\mathrm{Sh}(X, \mathbf{CBorn}_K)$, there might be the possibility to derive it in $\mathbf{LH}(X; \mathbf{CBorn}_K)$. We are going to discuss this in the following subsections.

3.3. **The category $\mathbf{LH}(\mathrm{Sh}(X, \mathbf{CBorn}_K))$.** Schneiders proves the following result about injective objects in the left heart of a quasi-abelian category.

Proposition 76 ([22, Proposition 2.1.15])**.** *Let* $\mathbf{E}$ *be a quasi-elementary quasi-abelian category. Then*

(i) *both the categories* $\mathbf{E}$ *and* $\mathbf{LH}(\mathbf{E})$ *are complete with exact products. Moreover,*

$$\mathbf{I} \colon \mathbf{E} \to \mathbf{LH}(\mathbf{E})$$

preserves projective limits.

(ii) *Both the categories* $\mathbf{E}$ *and* $\mathbf{LH}(\mathbf{E})$ *are cocomplete with strongly exact direct sums. Moreover,*

$$\mathbf{I} \colon \mathbf{E} \to \mathbf{LH}(\mathbf{E})$$

preserves direct sums.

(iii) *Both the categories* $\mathbf{E}$ *and* $\mathbf{LH}(\mathbf{E})$ *have enough projective objects. Moreover,* $\mathbf{LH}(\mathbf{E})$ *has enough injective objects.*

Beeing *quasi-elementary* is a technical property of a quasi-abelian category. We will not recall the precise definition and instead refer the reader to [22, Section 2.1]. However, we note that it is implied when the category is elementary. In particular, $\mathbf{CBorn}_K$ is quasi-elementary because of Lemma 15, and therefore the previous Proposition 76 implies that $\mathbf{LH}(\mathbf{CBorn}_K)$ has enough injectives.

This is good news, because one cannot expect $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ to be quasi-elementary. Luckily, we can write the left heart of $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ in terms of $\mathbf{LH}(\mathbf{CBorn}_K)$.

Proposition 77 ([22, Proposition 2.2.12])**.** *Let* $\mathbf{E}$ *be an elementary quasi-abelian category. The canonical inclusion*

$$\mathbf{I} \colon \mathbf{E} \to \mathbf{LH}(\mathbf{E})$$

gives a canonical functor

$$\mathbf{I} \colon \mathrm{Sh}(X, \mathbf{E}) \to \mathrm{Sh}(X, \mathbf{LH}(\mathbf{E})).$$

This functor induces an equivalence of categories

$$\mathbf{LH}(\mathrm{Sh}(X, \mathbf{E})) \simeq \mathrm{Sh}(X, \mathbf{LH}(\mathbf{E})).$$

In particular, we have the following

Corollary 78. *The category* $\mathbf{LH}(\mathrm{Sh}(X, \mathbf{CBorn}_K))$ *has enough injective objects.*

3.4. **The closed symmetric monoidal structure on $\mathbf{LH}(\mathbf{CBorn}_K)$.** We have shown in the previous subsection that the left heart of $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ has enough injectives. Now we can start thinking about a closed symmetric monoidal structure on $\mathrm{Sh}(X, \mathbf{LH}(\mathbf{CBorn}_K))$. In this subsection, we construct a closed symmetric monoidal structure on $\mathbf{LH}(\mathbf{CBorn}_K)$. The idea is as follows: One shows that the internal homomorphism functor as well as the internal tensor product on $\mathbf{CBorn}_K$ are derivable. Then one restricts these derived functors to the left heart.

We remark that basically all the hard work in this subsection has been done in [22, Section 1.5.2]. The author of this text merely showed that these general constructions of Schneiders apply to the category $\mathbf{CBorn}_K$.

Here is the result on the closed symmetric monoidal structure on $\mathbf{D}(\mathbf{CBorn}_K)$.

Proposition 79. *We have for any two objects P and I in $\mathbf{CBorn}_K$ that*

(i) $\underline{\mathrm{Hom}}_K(P, -)$ *is exact if P is projective,*
(ii) $\underline{\mathrm{Hom}}_K(-, I)$ *is exact if I is injective, and*
(iii) $\underline{\mathrm{Hom}}_K(P, I)$ *is injective if P is projective and I is injective.*

Moreover, the internal tensor product in $\mathbf{CBorn}_K$ is explicitly left derivable, the internal homomorphism functor in $\mathbf{CBorn}_K$ is explicitly right derivable, and we have the canonical functorial isomorphisms

(iv) $E \widehat{\otimes}^{\mathbb{L}}_K F \simeq F \widehat{\otimes}^{\mathbb{L}}_K E$,
(v) $E \widehat{\otimes}^{\mathbb{L}}_K K \simeq E \simeq K \widehat{\otimes}^{\mathbb{L}}_K E$,
(vi) $\mathbb{R}\,\mathrm{Hom}(E \widehat{\otimes}^{\mathbb{L}}_K F, G) \simeq \mathbb{R}\,\mathrm{Hom}(E, \mathbb{R}\,\underline{\mathrm{Hom}}_K(F, G))$, *and*
(vii) $\mathbb{R}\,\underline{\mathrm{Hom}}_K(K, G) \simeq G$,

where $E, F \in \mathbf{D}^-(\mathbf{CBorn}_K)$ and $G \in \mathbf{D}^+(\mathbf{CBorn}_K)$.

We now work towards a proof of this result.

Lemma 80. *Let P be a projective object in $\mathbf{CBorn}_K$. Then there exists a projective flat object $\tilde{P}$ in $\mathbf{CBorn}_K$ and morphisms*

$$\widetilde{P} \underset{e}{\overset{s}{\rightleftarrows}} P$$

such that $e \circ s = \mathrm{id}_P$, e is a strict epi, and s is a strict mono.

Proof. By Theorem 15, $\mathbf{CBorn}_K$ has enough flat projectives, so we can pick a flat projective object $\widetilde{P} \in \mathbf{CBorn}_K$ that admits a strict epimorphism

$$e \colon \widetilde{P} \to P.$$

Since P is projective, the existence of a right inverse s of e follows. The morphism s is a strict mono because of [22, Proposition 1.1.8]. □

Proposition 81. *Fix two complete convex bornological K-vector spaces P and P'. Assume that P is projective.*

(i) *Then the functor $P \widehat{\otimes}_K -$ is strongly exact.*
(ii) *There exists a full subcategory $\mathbf{P}$ of projectives in $\mathbf{CBorn}_K$ such that*
  – *any $E \in \mathbf{CBorn}_K$ admits a strict epimorphism $P \to E$ for some $P \in \mathbf{P}$ and*
  – *for any $P \in \mathbf{P}$, $P \widehat{\otimes}_K -$ is exact and preserves $\mathbf{P}$.*

Proof.

(i) The functor $P\widehat{\otimes}_K-$ is a right adjoint, and thus preserves cokernels. Therefore, it is strongly right exact. It remains to show that it is strongly left exact. To do so, pick an arbitrary strictly exact sequence

$$0 \longrightarrow E' \xrightarrow{u'} E \xrightarrow{u} E''.$$

We aim to show that the sequence

$$0 \longrightarrow P\widehat{\otimes}_K E' \xrightarrow{u'} P\widehat{\otimes}_K E \xrightarrow{u} P\widehat{\otimes}_K E''$$

is strictly exact.

Fix an object $\widetilde{P}$ as well as morphisms e and s as in Lemma 80. We get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & P\widehat{\otimes}_K E' & \xrightarrow{\mathrm{id}_P \widehat{\otimes}_K u'} & P\widehat{\otimes}_K E & \xrightarrow{\mathrm{id}_P \widehat{\otimes}_K u} & P\widehat{\otimes}_K E'' \\
 & & {\scriptstyle s\widehat{\otimes}_K \mathrm{id}_{E'}}\downarrow\ \uparrow{\scriptstyle e\widehat{\otimes}_K \mathrm{id}_{E'}} & & {\scriptstyle s\widehat{\otimes}_K \mathrm{id}_{E}}\downarrow\ \uparrow{\scriptstyle e\widehat{\otimes}_K \mathrm{id}_{E}} & & {\scriptstyle s\widehat{\otimes}_K \mathrm{id}_{E''}}\downarrow\ \uparrow{\scriptstyle e\widehat{\otimes}_K \mathrm{id}_{E''}} \\
0 & \longrightarrow & \widetilde{P}\widehat{\otimes}_K E' & \xrightarrow{\mathrm{id}_{\widetilde{P}} \widehat{\otimes}_K u'} & \widetilde{P}\widehat{\otimes}_K E & \xrightarrow{\mathrm{id}_{\widetilde{P}} \widehat{\otimes}_K u} & \widetilde{P}\widehat{\otimes}_K E'',
\end{array}$$

such that the row at the bottom is strictly exact. Also, note that the vertical morphisms pointing downwards are strict monomorphisms, and the vertical morphisms pointing upwards are strict epimorphisms, by [22, Proposition 1.1.8]

We now show that the row at the top of this diagram is strictly exact. First, we show the set-theoretic statement

$$(\mathrm{id}_P \widehat{\otimes}_K u')(P\widehat{\otimes}_K E') = \left\{x \in P\widehat{\otimes}_K E \colon (\mathrm{id}_P \widehat{\otimes}_K u)(x) = 0\right\}. \tag{3.2}$$

The inclusion $\subseteq$ is clear. To prove the converse, fix an element $x \in P\widehat{\otimes}_K E$ such that $(\mathrm{id}_P \widehat{\otimes}_K u)(x) = 0$. Because of the commutativity of the diagram above and because its row at the bottom is exact, there exists an element $x' \in \widetilde{P}\widehat{\otimes}_K E'$ such that

$$(\mathrm{id}_{\widetilde{P}} \widehat{\otimes}_K u')(x') = (s\widehat{\otimes}_K \mathrm{id}_E)(x).$$

Compute

$$\begin{aligned}
(\mathrm{id}_P \widehat{\otimes}_K u')((e\widehat{\otimes}_K \mathrm{id}_{E'})(x')) &= (e\widehat{\otimes}_K \mathrm{id}_E)((\mathrm{id}_{\widetilde{P}} \widehat{\otimes}_K u')(x')) \\
&= (e\widehat{\otimes}_K \mathrm{id}_E)((s\widehat{\otimes}_K \mathrm{id}_E)(x)) \\
&= ((e \circ s)\widehat{\otimes}_K \mathrm{id}_E)(x) \\
&= (\mathrm{id}_P \widehat{\otimes}_K \mathrm{id}_E)(x) \\
&= x.
\end{aligned}$$

Since $(e\widehat{\otimes}_K \mathrm{id}_{E'})(x') \in P\widehat{\otimes}_K E'$, this shows the desired inclusion $\supseteq$.

Finally, we have to show that $\mathrm{id}_P \widehat{\otimes}_K u'$ is strict. This follows, because $e\widehat{\otimes}_K \mathrm{id}_E$ is a strict epimorphism, $\mathrm{id}_{\widetilde{P}} \widehat{\otimes}_K u'$ and $s\widehat{\otimes} \mathrm{id}_{E'}$ are strict monomorphisms, and

$$\mathrm{id}_P \widehat{\otimes}_K u' = \left(e\widehat{\otimes}_K \mathrm{id}_E\right) \circ \left(\mathrm{id}_{\widetilde{P}} \widehat{\otimes}_K u'\right) \circ \left(s\widehat{\otimes} \mathrm{id}_{E'}\right).$$

(ii) By [32, Theorem 3.50], $E\widehat{\otimes}_K -$ is exact for every complete convex bornological K-vector space E. Furthermore, we can choose $\mathbf{P}$ to be the full subcategory of $\mathbf{CBorn}_K$ of all direct sums of c_0-*spaces*. We refer the reader to [19, Proof of Lemma 3.14] for the precise definition of these spaces, and the proof of (ii) of our Proposition 81.

□

Now the...

Proof of Proposition 79. ...is easy: We aim to apply [22, Proposition 1.5.3]. However, the conditions for the Proposition in the article [22] are actually to strong. The results from the previous Proposition 81 are enough to make the [22, Proof of Proposition 1.5.3] work. We remark that this observation is not due to the author, it has been made in [19, Lemma 2.4 and the remark following it]. □

As $\mathbf{LH}(\mathbf{CBorn}_K)$ is a full subcategory of $\mathbf{D}(\mathbf{CBorn}_K)$, we get a closed symmetric monoidal structure on the left heart by restricting the one from the derived category on it.

Corollary 82. *The category* $\mathbf{LH}(\mathbf{CBorn}_K)$ *is canonically a closed abelian category. Its internal tensor product is given by*

$$-\widehat{\otimes}_{\mathbf{I}(K)}- = LH^0 \circ \left(-\widehat{\otimes}^{\mathbb{L}}_K-\right) \colon \mathbf{LH}(\mathbf{CBorn}_K) \times \mathbf{LH}(\mathbf{CBorn}_K) \to \mathbf{LH}(\mathbf{CBorn}_K),$$

its unit object by $\mathbf{I}(K)$*, and its internal homomorphisms functor by*

$$\underline{\mathrm{Hom}}_{\mathbf{I}(K)} = LH^0 \circ \mathbb{R}\,\underline{\mathrm{Hom}}_K \colon \mathbf{LH}(\mathbf{CBorn}_K)^{\mathrm{op}} \times \mathbf{LH}(\mathbf{CBorn}_K) \to \mathbf{LH}(\mathbf{CBorn}_K).$$

The functor $-\widehat{\otimes}_{\mathbf{I}(K)}-$ *is explicitly left derivable and the functor* $\underline{\mathrm{Hom}}_{\mathbf{I}(K)}$ *is explicitly right darivable. Furthermore, we have canonical isomorphisms*

$$\mathbf{I}(E)\widehat{\otimes}^{\mathbb{L}}_{\mathbf{I}(K)}\mathbf{I}(F) \simeq E\widehat{\otimes}^{\mathbb{L}}_K F$$

and

$$\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(\mathbf{I}(F), \mathbf{I}(G)) \simeq \mathbb{R}\,\underline{\mathrm{Hom}}_K(F, G)$$

for any $E, F \in \mathbf{D}^-(\mathbf{CBorn}_K)$ *and* $G \in \mathbf{D}^+(\mathbf{CBorn}_K)$*.*

Also, for any projective object Q *of* $\mathbf{LH}(\mathbf{CBorn}_K)$*, the functor*

$$Q\widehat{\otimes}_{\mathbf{I}(K)}- \colon \mathbf{LH}(\mathbf{CBorn}_K) \to \mathbf{LH}(\mathbf{CBorn}_K)$$

is exact and $Q\widehat{\otimes}_{\mathbf{I}(K)}Q'$ *is projective if* Q' *is projective in* $\mathbf{LH}(\mathbf{CBorn}_K)$*.*

Proof. As in the proof of Proposition 79, the conditions proven in Proposition 81 are enough to make the [22, Proof of Corollary 1.5.4] work. This observation is again due to [19, Lemma 2.4 and the remark following it]. □

We close this subsection with the following result, which gives a relationship between the closed symmetric monoidal structures on $\mathbf{CBorn}_K$ and $\mathbf{LH}(\mathbf{CBorn}_K)$ via the functor $\mathbf{I}$.

Proposition 83. *Let* E *and* F *denote two complete convex bornological* K*-vector spaces. Then we have a natural isomorphism*

$$\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(\mathbf{I}(E), \mathbf{I}(F)) \simeq \mathbf{I}(\underline{\mathrm{Hom}}_K(E, F)).$$

The proof of this result relies on the following

Lemma 84. *For every complete convex bornological K-vector space F, the functor*

$$\underline{\mathrm{Hom}}_K(-,F)\colon \mathbf{CBorn}_K^{\mathrm{op}} \to \mathbf{CBorn}_K$$

is left exact.

Proof. Fix a strictly exact sequence

$$0 \longrightarrow E' \xrightarrow{u'} E \xrightarrow{u} E'' \longrightarrow 0.$$

in $\mathbf{CBorn}_K$. We have to show that

$$0 \longrightarrow \underline{\mathrm{Hom}}_K(E'',F) \xrightarrow{u^*} \underline{\mathrm{Hom}}_K(E,F) \xrightarrow{u'^*} \underline{\mathrm{Hom}}_K(E',F)$$

is strictly exact. We proceed in two steps.

Step 1. We aim to show that

$$\underline{\mathrm{Hom}}_K(E'',F) \xrightarrow{u^*} \underline{\mathrm{Hom}}_K(E,F) \xrightarrow{u'^*} \underline{\mathrm{Hom}}_K(E',F)$$

is strictly exact.

First, we will show that $u^*\left(\underline{\mathrm{Hom}}_K(E'',F)\right) = \ker(u'^*)$. The inclusion $\subseteq$ is clear. To prove the converse, pick a bounded linear map $f \in \ker(u'^*)$. Then we define a linear map $f'' \in \underline{\mathrm{Hom}}_K(E'',F)$ via

$$f''(e'') := f(e) \text{ for any } e \in u^{-1}(e'').$$

This is well-defined: First, notice that u is surjective, so $u^{-1}(e'') \neq \emptyset$ for any $e'' \in E''$. Next, suppose that $e_1, e_2 \in u^{-1}(e'')$. Then we have

$$e_1 - e_2 \in \ker(u) = \mathrm{im}(u') \subseteq \ker(f)$$

and thus

$$f(e_1) - f(e_2) = f(e_1 - e_2) = 0.$$

This shows that f'' is well-defined. Now we show that $f'' \in \underline{\mathrm{Hom}}_K(E'',F)$. That is, we have to show that f'' is bounded. To prove this, pick an arbitrary bounded subset $B'' \subseteq E''$. Since u is a strict epimorphism, there exists a bounded subset $B \subseteq E$ such that $u(B) = B''$. Then $f''(B'') = f(u(B))$ is bounded since $f \circ u$ is bounded. Because the choice of B'' was arbitrary, it follows that f'' is bounded. This proves the claim that $u^*\left(\underline{\mathrm{Hom}}_K(E'',F)\right) = \ker(u'^*)$.

Recall that the inclusion $\ker(u'^*) \hookrightarrow \underline{\mathrm{Hom}}_K(E,F)$ is a strict monomorphism by [22, Remark 1.1.2]. It follows with Lemma 18 (iii) that $\ker(u'^*)$ is a closed subspace of $\underline{\mathrm{Hom}}_K(E,F)$ and we compute

$$\mathrm{im}(u^*) = \overline{u^*\left(\underline{\mathrm{Hom}}_K(E'',F)\right)} = \overline{\ker(u'^*)} = \ker(u').$$

Step 2. We aim to show that

$$0 \longrightarrow \underline{\mathrm{Hom}}_K(E'',F) \xrightarrow{u^*} \underline{\mathrm{Hom}}_K(E,F)$$

is strictly exact. That is, we aim to show that u^* is a strict monomorphism.

By Lemma 18, it remains to show that u^* is injective and the bornology on $\underline{\mathrm{Hom}}_K(E'',F)$ is the subspace bornology induced by the bornology on $\underline{\mathrm{Hom}}_K(E,F)$. First, we show the injectivity. Let $f \in \underline{\mathrm{Hom}}_K(E'',F)$ such that $f \circ u = u^*(f) = 0$, and fix an arbitary $e'' \in E''$. Since u is surjective, there exists an $e \in u^{-1}(e'')$, and thus $f(e'') = f(u(e)) = 0$. Since the choice of $e'' \in E''$ was arbitrary, it follows that $f = 0$, and thus u^* is injective. Next, we have to show that for any bounded subset $\Phi \subseteq \underline{\mathrm{Hom}}_K(E,F)$, $(u^*)^{-1}(\Phi)$ is bounded. To do so, pick an arbitrary bounded subset $B'' \subseteq E''$. Since u is strict, there exists a bounded subset $B \subseteq E$ such that $u(B) = B''$. Now we compute

$$\begin{aligned}\left((u^*)^{-1}(\Phi)\right)(B'') &= \bigcup_{f''\in(u^*)^{-1}(\Phi)} f''(B'') = \bigcup_{f''\in(u^*)^{-1}(\Phi)} f''(u(B)) \\ &\subseteq \bigcup_{f\in\Phi} f(B) = \Phi(B).\end{aligned}$$

Since Φ is bounded, it follows that $\Phi(B)$ and therefore $\left((u^*)^{-1}(\Phi)\right)(B'')$ is bounded. And because the choice of B'' was arbitrary, it follows that $(u^*)^{-1}(\Phi)$ is bouded. Thus u^* is a strict monomorphism.

This finishes the proof of Lemma 84. □

Proof of Proposition 83. Pick a projective resolution

$$\ldots \to P^{-2} \to P^{-1} \to P^0 \to E \to 0.$$

and compute

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(\mathbf{I}(E),\mathbf{I}(F)) \\ &= LH^0\left(\mathbb{R}\,\underline{\mathrm{Hom}}_K(E,F)\right) \\ &= LH^0\left(0 \to \underline{\mathrm{Hom}}_K(P^0,F) \to \underline{\mathrm{Hom}}_K(P^{-1},F) \to \ldots\right) \\ &= \left(0 \to \operatorname{coim}\left(0 \to \underline{\mathrm{Hom}}_K(P^0,F)\right) \to \ker\left(\underline{\mathrm{Hom}}_K(P^0,F) \to \underline{\mathrm{Hom}}_K(P^{-1},F)\right) \to 0\right) \\ &\overset{84}{=} \left(0 \to 0 \to \underline{\mathrm{Hom}}_K(E,F) \to 0\right) \\ &= \mathbf{I}(\underline{\mathrm{Hom}}_K(E,F)).\end{aligned}$$

□

3.5. **The closed symmetric monoidal structure on $\mathbf{LH}(\mathrm{Sh}(X,\mathbf{CBorn}_K))$.** Having a closed structure on $\mathbf{LH}(\mathbf{CBorn}_K)$ enables us now to define a closed symmetric monoidal structure on $\mathrm{Sh}(X,\mathbf{LH}(\mathbf{CBorn}_K))$. In this subsection, we briefly summarize this construction, following [22, Section 2.2.3].

First, let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves with values in $\mathbf{LH}(\mathbf{CBorn}_K)$. Then we define the object $\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(\mathcal{F},\mathcal{G})$ in $\mathbf{LH}(\mathbf{CBorn}_K)$ to be the kernel of the morphism

$$r\colon \prod_{U\in X_w} \underline{\mathrm{Hom}}_{\mathbf{I}(K)}(\mathcal{F}(U),\mathcal{G}(U)) \to \prod_{\substack{V,W\in X_w\\ W\subseteq V}} \underline{\mathrm{Hom}}_{\mathbf{I}(K)}(\mathcal{F}(V),\mathcal{G}(W)),$$

which is defined by

$$r(f)_{V,W} := \sigma_{W\subseteq V}\circ f_V - f_W \circ \tau_{W\subseteq V}.$$

Here $\sigma_{W\subseteq V}\colon \mathcal{F}(V)\to\mathcal{F}(W)$ and $\tau_{W\subseteq V}\colon \mathcal{G}(V)\to\mathcal{G}(W)$ are the restriction maps.

By [22, Proposition 2.2.14], we now get an object

$$\underline{\mathcal{H}om}_{\mathbf{I}(K)}(\mathcal{F},\mathcal{G})(U) := \underline{\mathrm{Hom}}_{\mathbf{I}(K)}(\mathcal{F}|_U,\mathcal{G}|_U)$$

in $\mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K))$.

Next, we define the presheaf

$$\mathcal{F}\widehat{\otimes}_{\mathbf{I}(K),\mathrm{Psh}}\mathcal{G} : U \mapsto \mathcal{F}(U)\widehat{\otimes}_{\mathbf{I}(K)}\mathcal{G}(U),$$

as well as its sheafification

$$\mathcal{F}\widehat{\otimes}_{\mathbf{I}(K)}\mathcal{G} := \left(\mathcal{F}\widehat{\otimes}_{\mathbf{I}(K),\mathrm{Psh}}\mathcal{G}\right)^{\mathrm{Sh}}.$$

Proposition 85 ([22, Corollary 2.2.17])**.** *The category* $\mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K))$ *endowed with* $-\widehat{\otimes}_{\mathbf{I}(K)}-$ *as internal tensor product,* $\mathbf{I}(K)_X$ *as unit, and* $\underline{\mathcal{H}om}_{\mathbf{I}(K)}$ *as internal homomorphism functor is a closed abelian category.*

3.6. **The closed symmetric monoidal structure on $\mathbf{D}(\mathrm{Sh}(X,\mathbf{CBorn}_K))$.** In this subsection, we finally discuss how to derive the internal tensor product, as well as the internal homomorphism functor, using the techniques from [22, Section 2.3.2].

First, we establish the following

Lemma 86. *Let* $Q \in \mathbf{LH}(\mathbf{CBorn}_K)$ *be a projective object.*

(i) *Then* $\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(Q,-)$ *is exact, and*
(ii) $Q\widehat{\otimes}_{\mathbf{I}(K)}-$ *is exact as well.*

Proof.

(i) Fix a short exact sequence

$$0 \longrightarrow E' \xrightarrow{u'} E \xrightarrow{u} E'' \longrightarrow 0$$

in $\mathbf{LH}(\mathbf{CBorn}_K)$. We aim to show that

$$0 \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{I}(K)}(Q,E') \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{I}(K)}(Q,E) \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{I}(K)}(Q,E'') \longrightarrow 0$$

is exact. Since $\mathbf{LH}(\mathbf{CBorn}_K)$ is elementary, it is cocomplete and has a small strictly generating set $\mathcal{G}$ of tiny projective objects. Thus we have to show, by [22, Proposition], that the sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathbf{I}(K)}(G,\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(Q,E')) \to \mathrm{Hom}_{\mathbf{I}(K)}(G,\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(Q,E)) \to \mathrm{Hom}_{\mathbf{I}(K)}(G,\underline{\mathrm{Hom}}_{\mathbf{I}(K)}(Q,E'')) \to 0$$

is exact for any $G \in \mathcal{G}$. But this sequence is isomorphic to

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{I}(K)}(G\widehat{\otimes}_{\mathbf{I}(K)}Q,E') \longrightarrow \mathrm{Hom}_{\mathbf{I}(K)}(G\widehat{\otimes}_{\mathbf{I}(K)}Q,E) \longrightarrow \mathrm{Hom}_{\mathbf{I}(K)}(G\widehat{\otimes}_{\mathbf{I}(K)}Q,E'') \longrightarrow 0,$$

which is exact, because $G\widehat{\otimes}_{\mathbf{I}(K)}Q$ is projective by Corollary 82.

(ii) is a statement in Corollary 82.

□

The previous result shows that the discussion from [22, Section 2.3.2] applies to our setting. A reformulation of [22, Proposition 2.3.10] now gives the following result.

Proposition 87. *The functor*

$$-\widehat{\otimes}_{\mathbf{I}(K)}-\colon \mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K))\times\mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K)) \to \mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K))$$

is explicitly left derivable and the functor

$$\underline{\mathcal{H}om}_{\mathbf{I}(K)}\colon \mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K))^{\mathrm{op}}\times\mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K)) \to \mathrm{Sh}(X;\mathbf{LH}(\mathbf{CBorn}_K))$$

is explicitly right derivable. Moreover, we have the canonical functorial isomorphisms

(i) $E\widehat{\otimes}^{\mathbb{L}}_{\mathbf{I}(K)}F \simeq F\widehat{\otimes}^{\mathbb{L}}_{\mathbf{I}(K)}E,$

(ii) $\underline{\mathbf{I}(K)}_X \widehat{\otimes}^{\mathbb{L}}_{\mathbf{I}(K)} E \simeq E$,
(iii) $\mathbb{R}\,\underline{\mathcal{H}om}_{\mathbf{I}(X)}(E\widehat{\otimes}^{\mathbb{L}}_{\mathbf{I}(K)} F, G) \simeq \mathbb{R}\,\underline{\mathcal{H}om}_{\mathbf{I}(X)}(E, \mathbb{R}\,\underline{\mathcal{H}om}_{\mathbf{I}(K)}(F,G))$, *and*
(iv) $\mathbb{R}\,\underline{\mathcal{H}om}_{\mathbf{I}(K)}(\underline{\mathbf{I}(K)}_X, E)) \simeq E$.

We close this subsection with a discussion on the relation between the internal homomorphism functors $\underline{\mathcal{H}om}_K$ on $\mathrm{Sh}(X, \mathbf{CBorn}_K)$ and $\underline{\mathcal{H}om}_{\mathbf{I}(K)}$ on $\mathrm{Sh}(X; \mathbf{LH}(\mathbf{CBorn}_K))$.

Proposition 88. *We have for any two sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X, \mathbf{CBorn}_K)$ a natural isomorphism*

$$\underline{\mathcal{H}om}_{\mathbf{I}(K)}\left(\mathbf{I}(\mathcal{F}), \mathbf{I}(\mathcal{G})\right) = \mathbf{I}\left(\underline{\mathcal{H}om}_K\left(\mathcal{F}, \mathcal{G}\right)\right).$$

Proof. We have to show that there is a natural isomorphism

$$\underline{\mathrm{Hom}}_{\mathbf{I}(K)}\left(\mathbf{I}(\mathcal{F})|_U, \mathbf{I}(\mathcal{G})|_U\right) = \mathbf{I}\left(\underline{\mathrm{Hom}}_K\left(\mathcal{F}|_U, \mathcal{G}|_U\right)\right).$$

for every admissible open $U \in X_w$. Recall that $\underline{\mathrm{Hom}}_{\mathbf{I}(K)}$ and $\underline{\mathrm{Hom}}_K$ were defined similarly, in terms of products and equalizers. Since $\mathbf{I}$ is a right adjoint its commutes with limits, and thus the lemma follows with Proposition 83. □

3.7. **Derived bounded linear endomorphisms of rigid analytic varieties.** Having all this new technical machinery at hand, we can revisit our initial question 34. Note that Proposition 88 implies

$$\begin{aligned}\mathbb{R}^0\,\underline{\mathcal{H}om}_{\mathbf{I}(K)}\left(\mathbf{I}\left(\mathcal{O}_X\right), \mathbf{I}\left(\mathcal{O}_X\right)\right) &= \underline{\mathcal{H}om}_{\mathbf{I}(K)}\left(\mathbf{I}\left(\mathcal{O}_X\right), \mathbf{I}\left(\mathcal{O}_X\right)\right)\\ &= \mathbf{I}\left(\underline{\mathcal{H}om}_K\left(\mathcal{O}_X, \mathcal{O}_X\right)\right),\end{aligned}$$

and therefore ρ_X gives rise to a morphism

$$\mathrm{R}_X \colon \mathbf{I}\left(\widehat{\mathcal{D}}_X\right) \to \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(K)}\left(\mathbf{I}\left(\mathcal{O}_X\right), \mathbf{I}\left(\mathcal{O}_X\right)\right).$$

in $\mathbf{D}(\mathrm{Sh}(X, \mathbf{CBorn}_K))$. We can now state a precise formulation of question 34.

Question 89. *Is the morphism*

$$\mathrm{R}_X \colon \mathbf{I}\left(\widehat{\mathcal{D}}_X\right) \to \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(K)}\left(\mathbf{I}\left(\mathcal{O}_X\right), \mathbf{I}\left(\mathcal{O}_X\right)\right).$$

an isomorphism in the derived category of sheaves of bornological K-vector spaces $\mathbf{D}(Sh(X, \mathbf{CBorn}_K))$? Equivalently, do we have

$$\underline{\mathcal{E}xt}^i_{\mathbf{I}(K)}\left(\mathbf{I}\left(\mathcal{O}_X\right), \mathbf{I}\left(\mathcal{O}_X\right)\right) = 0 \text{ for all } i > 0?$$

The author aims to attack this question via establishing a derived version of Theorem 66. The main difficulty is that $\mathbf{I}$ is a *right* adjoint. Therefore, it does not commute with colimits, it does not commute with sheafification, and it does not commute with the monoidal structures. In particular, it is currently not clear to the author how $\mathbf{I}(\mathcal{F})\widehat{\otimes}_{\mathbf{I}(K)}\mathbf{I}(\mathcal{G})$ relates to $\mathbf{I}\left(\mathcal{F}\widehat{\otimes}_K \mathcal{G}\right)$ for two complete convex bornological sheaves $\mathcal{F}$ and $\mathcal{G}$. Also, recall from Section 2.1, or to be precise diagram 2.1, that the definition of the internal homomorphism functor $\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}$ relies on the monoidal structure in $\mathbf{LH}(\mathrm{Sh}(X, \mathbf{CBorn}_K))$. This makes it even more difficult to investigate the $\mathbf{I}(\mathcal{O}_X)$-dual of $\mathbf{I}(\varphi_X)$.

This is why we close this section with a

Conjecture 90. *There exist morphisms*

$$\widetilde{R}_X \colon \mathbf{I}\left(\widehat{\mathcal{D}}_X\right) \to \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}\left(\mathbf{I}(\Delta^{-1}\mathcal{O}_{X\times X}), \mathbf{I}(\mathcal{O}_X)\right)$$

and

$$Z_X\colon \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(K)}(\mathbf{I}(\mathcal{O}_X),\mathbf{I}(\mathcal{O}_X)) \to \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}\left(\mathbf{I}\left(\mathcal{O}_X\widehat{\otimes}_K\mathcal{O}_X\right),\mathbf{I}(\mathcal{O}_X)\right)$$

of $\mathbf{I}(\mathcal{O}_X)$*-module objects such that the following statements are true.*

(i) *All the previously defined moprhisms fit into a commutative diagram*

(3.3)

$$\begin{array}{ccc} \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}(\mathbf{I}(\Delta^{-1}\mathcal{O}_{X\times X}),\mathbf{I}(\mathcal{O}_X)) & \xrightarrow{\mathbf{I}(\varphi_X)^*} & \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}(\mathbf{I}\left(\mathcal{O}_X\widehat{\otimes}_K\mathcal{O}_X\right),\mathbf{I}(\mathcal{O}_X)) \\ \uparrow{\scriptstyle \widetilde{\mathrm{R}}_X} & & \uparrow{\scriptstyle \mathrm{Z}_X} \\ \mathbf{I}(\widehat{\mathcal{D}}_X) & \xrightarrow{\mathrm{R}_X} & \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(K)}(\mathbf{I}(\mathcal{O}_X),\mathbf{I}(\mathcal{O}_X)) \end{array}$$

of $\mathbf{I}(\mathcal{O}_X)$*-module objects. Furthermore,* Z_X *is an isomorphism.*

(ii) *Let* K *be algebraically closed. Then* $\widetilde{\mathrm{R}}_X$ *is an isomorphism too. In particular,* R_X *is the* $\mathbf{I}(\mathcal{O}_X)$*-dual of* $\mathbf{I}(\varphi_X)$*.*

4. Outlook

In the last section of this text, we give a brief discussion of the research that the author aims to do next.

Clearly, we aim to prove Conjecture 90. Once this Conjecture is established, Question 89 reduces to the following two questions.

(i) Is the object
$$\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}\left(\mathbf{I}(\Delta^{-1}\mathcal{O}_{X\times X}),\mathbf{I}(\mathcal{O}_X)\right)$$
concentrated in degree zero if K is algebraically closed?

(ii) Is the morphism $\mathbf{I}(\varphi_X)^*$ an isomorphism if the ground field K is algebraically closed and its residue field k is uncountable?

The author believes that answering (i) and (ii) is in reach! Let us have a look on possible approaches.

We first consider (i). For simplicity, assume that $X=\mathbb{D}^1=\operatorname{Sp}K\langle x\rangle$ is the one-dimensional unit disc over K. Denote the zero section of the projection

$$\pi\colon T^*\mathbb{D}^1\to\mathbb{D}^1$$

by s. One can show that

$$s^{-1}\mathcal{O}_{T^*\mathbb{D}^1}\simeq\Delta^{-1}\mathcal{O}_{\mathbb{D}^2}$$

and find a morphism

$$\tau_{\mathcal{F}}\colon \pi_*\mathcal{F}\to\underline{\mathcal{H}om}_{\mathcal{O}_{\mathbb{D}^1}}\left(s^{-1}\mathcal{O}_{T^*\mathbb{D}^1},\mathcal{F}\right)$$

for every complete convex bornological $\mathcal{O}_{T^*\mathbb{D}^1}$-module $\mathcal{F}$. The author aims to show that this morphism admits an analog in $\mathbf{LH}(\mathrm{Sh}(X,\mathbf{CBorn}_K))$, which would then give rise to a morphism

$$\mathrm{T}_{\mathbb{D}^1}\colon \mathbb{R}\pi_*\mathbf{I}\left(\mathcal{O}_{T^*\mathbb{D}^1}\right)\to\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}\left(\mathbf{I}(\Delta^{-1}\mathcal{O}_{\mathbb{D}^2}),\mathbf{I}(\mathcal{O}_{\mathbb{D}^1})\right)$$

in $\mathbf{D}(\mathrm{Sh}(X,\mathbf{CBorn}_K))$. If $\mathrm{T}_{\mathbb{D}^1}$ is an isomorphism, then this would reduce question (i) to a calculation of $\mathbb{R}\pi_*\mathbf{I}\left(\mathcal{O}_{T^*\mathbb{D}^1}\right)$. Note that for every $i\geq 0$, the sheaf $\mathbb{R}^i\pi_*\mathbf{I}\left(\mathcal{O}_{T^*\mathbb{D}^1}\right)$ is the sheafification of the presheaf

$$U\mapsto H^i\left(\mathbb{A}^{1,\mathrm{an}}\times U,\mathbf{I}\left(\mathcal{O}_{\mathbb{A}^{1,\mathrm{an}}\times U}\right)\right),$$

see for example [33, Proposition III.8.1]. $\mathbb{A}^{1,\mathrm{an}} \times U$ is a *relative Stein-space*, and in particular *quasi-Stein*, by [34, Section 2.1]. Thus we could prove the vanishing of $\mathbb{R}^i\pi_*\mathbf{I}\left(\mathcal{O}_{T^*\mathbb{D}^1}\right)$ for $i > 0$ through a version of Kiehl's Theorem B [35, Satz 2.4.2] for sheaves with values in $\mathbf{LH}(\mathbf{CBorn}_K)$. That is, we have to show that the Čech complex in Kiehl's Theorem B is *strictly* exact. The author believes that this should follow directly from Kiehl's result, in conjunction with *Buchwalter's theorem* [25, Theorem 4.9], which is a version of the open mapping theorem for complete convex bornological vector spaces with countable basis.

If this approach works, it would give a positive answer to (i). However, it might be more difficult to give an answer to (ii). Since we are not having a good description of $\underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}$ yet, we continue this discussion in the category $\mathrm{Sh}(X, \mathbf{CBorn}_K)$. We are interested in morphism of the form $\varphi_X^{\mathcal{F}} = \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\varphi_X, \mathcal{F}\right)$, that is

$$\varphi_X^{\mathcal{F}} \colon \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{F}\right) \to \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X, \mathcal{F}\right),$$

where $\mathcal{F}$ is some complete convex bornological $\mathcal{O}_X$-module.

We have the following generalisation of the notion of a rapidly decreasing sequence in a K-Banach space.

Definition 91. *Let E denote a complete convex bornological K-vector space. A sequence $(a_\alpha)_{\alpha\in\mathbb{N}^d} \subseteq E$ is* rapidly decreasing *if for every $r \in \mathbb{N}$ the set*

$$\left\{\pi^{-r|\alpha|}a_\alpha \colon \alpha \in \mathbb{N}^d\right\} \subseteq E$$

is bounded.

We have the following local criterion.

Lemma 92. *Let X be affinoid and equipped with an étale morphism $g\colon X \to \mathbb{D}^d$. For any sheaf $\mathcal{F}$ of complete convex bornological $\mathcal{O}_X$-modules, the following two statements are equivalent.*

(i) *The morphism*

$$\varphi_X^{\mathcal{F}} \colon \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{F}\right) \to \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X, \mathcal{F}\right)$$

is an isomorphism of complete convex bornological $\mathcal{O}_X$-modules.

(ii) *For any $\phi \in \underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X, \mathcal{F}\right)$, the family*

$$\left(\varphi(X)\left((y-x)^\alpha\right)\right)_{\alpha\in\mathbb{N}} \subseteq \mathcal{F}(X)$$

is rapidly decreasing.

Proof. The direction (i)⇒(ii) follows because

$$\left((y-x)^\alpha\right)_{\alpha\in\mathbb{N}} \subseteq \left(\Delta^{-1}\mathcal{O}_{X\times X}\right)(X)$$

is rapidly decreasing. The proof of (ii)⇒(i) is completely analogous to the proof of Proposition 57. First note that since for every $U \in X_w$ the composition $U \hookrightarrow X \xrightarrow{g} \mathbb{D}^d$ is again étale and

$$\left(\varphi(U)\left((y-x)^\alpha\right)\right)_{\alpha\in\mathbb{N}} = \left(\varphi(X)\left((y-x)^\alpha\right)|_U\right)_{\alpha\in\mathbb{N}} \subseteq \mathcal{F}(U)$$

again rapdily decreasing, it is enough to show that

$$\varphi_X^{\mathcal{F}}(X) \colon \underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\Delta^{-1}\mathcal{O}_{X\times X}, \mathcal{F}\right) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X, \mathcal{F}\right)$$
$$\phi \mapsto \phi \circ \varphi$$

is an isomorphism of complete convex bornological $\mathcal{O}_X(X)$-modules. Now suppose that (ii) of Lemma 92 is satisfied. Then the proof of Proposition 57(i) says that $\varphi_X^{\mathcal{F}}(X)$ is an isomorphism of algebraic $\mathcal{O}_X(X)$-modules, and Proposition 57(ii) implies that $\varphi_X^{\mathcal{F}}(X)$ is strict. That is, it is indeed an isomorphism of complete convex bornological $\mathcal{O}_X$-modules. □

The previous Lemma 92 makes very concrete calculations possible. For example, Ardakov-Ben-Bassat's proof of their Theorem 30 actually uses the previous Lemma: They show in their [28, Theorem 4.9] that for an affinoid X equipped with an étale morphism $g\colon X \to \mathbb{D}^d$, $\mathcal{O}_X$ satisfies (ii) of Lemma 92 if the residue field k of K is algebraically closed and uncountable.

The author hopes to find an analog of Lemma 92 for morphisms of the form $\mathbf{I}(\varphi_X)^{\mathcal{F}} = \underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}(\mathbf{I}(\varphi_X), \mathcal{F})$, that is

$$\mathbf{I}(\varphi_X)^{\mathcal{F}} : \underline{\mathcal{H}om}_{\mathbf{I}(\mathcal{O}_X)}\left(\mathbf{I}(\Delta^{-1}\mathcal{O}_{X\times X}), \mathcal{F}\right) \to \underline{\mathcal{H}om}_{\mathcal{O}_X}\left(\mathbf{I}\left(\mathcal{O}_X \widehat{\otimes}_K \mathcal{O}_X\right), \mathcal{F}\right),$$

where $\mathcal{F}$ denotes an $\mathbf{I}(\mathcal{O}_X)$-module object. Then he hopes to be able to investigate morphisms $\varphi_X^{\mathcal{I}}$, where $\mathcal{I}$ denotes an injective object in the category of $\mathbf{I}(\mathcal{O}_X)$-module objects $\mathbf{Mod}(\mathbf{I}(\mathcal{O}_X))$. Hopefully, this is sufficient for understanding whether the morphism $\mathbf{I}(\varphi_X)^*$ is an isomorphism if the ground field K is algebraically closed and its residue field k is uncountable.

Once Conjecture 90 is proven to be true, and the answers to both questions (i) and (ii) are yes, it would follow that the morphism

$$\mathrm{R}_X \colon \mathbf{I}\left(\widehat{\mathcal{D}}_X\right) \to \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{I}(K)}\left(\mathbf{I}(\mathcal{O}_X), \mathbf{I}(\mathcal{O}_X)\right)$$

is an isomorphism in the derived category of sheaves of bornological K-vector spaces $\mathbf{D}(Sh(X, \mathbf{CBorn}_K))$. Then a Riemann-Hilbert correspondence for $\widehat{\mathcal{D}}$-modules would be in reach, as described in Subsection 1.6.

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